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Condensation of Singularities
of a
Function of One Variable

Mathematics and Physics
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THE CONDENSATION OF SINGULARITIES

OF

A FUNCTION OF ONE VARIABLE.

BY PERCY ALMERIN SMITH.

THESIS FOR THE DEGREE OF BACHELOR OF SCIENCE IN

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. Percy A. Smith.

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The Condensation of Singularities.

Introduction.

The object of this thesis is to set forth methods for building up from functions which are discontinuous at a single point or have a single point without a derivative, functions which shall have such singularities at a certain set of points, either at finite distances from each other or everywhere dense.

Chapter I deals with discontinuities only. The functions discussed in this chapter will be treated according to the methods devised by Hankel and Cantor. In addition to this the expression

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin m\pi x)}{m^2}$$

will be taken up and treated in the same manner as

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin n\pi x)}{n^s}$$

is treated by Hamel. There will also be a discussion of a method by which one such function is defined by a single term instead of a series, namely;

$$f(x) = \varphi(\sin 1^n \pi x)$$

and

$$f(x) = \lim_{n=\infty} \varphi(\sin 1^n \pi x)$$

and lastly

$$f(x) = \lim_{n=\infty} x\, \varphi(\sin 1^n \pi x)$$

At the end of each section problems will be given illustrating the method set forth in that section. This chapter will also contain a comparison of the methods of Hamel and Cantor.

Chapter II deals with the application of the methods of Hamel and Cantor to functions having no derivatives at a single point, but which are continuous at that point.

Chapter III contains problems illustrating the methods set forth in chapters I and II.

Chapter I.

Discontinuities.

Section I.

In this section we shall treat our function according to the method of *Hankel, except that we shall use the substitution

$$y = \sin \lfloor n \rfloor \pi x$$

instead of

$$y = \sin n \pi x$$

and in the denominator we shall use $\lfloor n \rfloor$ instead of n^s.

Let $\varphi(y)$ be a function of y which, for all values of y between -1 and $+1$, the end points included, is finite and determined, and moreover, is continuous, except at the point $y = 0$, and is equal to zero for $y = 0$.

Our problem now is to build up from this function, $\varphi(y)$, which is "discontinuous" at the point $y = 0$, a new function, $f(x)$, which shall be discon-

* See Dini, page 157, § 107.

tinuous at a set of points every where dense within any given finite interval, say for all rational points.

First let us examine the series

$$F(y) \equiv \sum_{n=1}^{n=\infty} \frac{\varphi(y)}{\lfloor n}$$

This new function of y will obey the same laws as to continuity or discontinuity as our original function $\varphi(y)$, for we may write it as follows:

$$F(y) = \varphi(y) \sum_{n=1}^{\infty} \frac{1}{\lfloor n}$$

$$= (e-1)\,\varphi(y)$$

since

$$(e-1) = \sum_{n=1}^{\infty} \frac{1}{\lfloor n}$$

(See Bowser's Coll. Alg. p. 444)

That is, $F(y)$ is equal to our original function $\varphi(y)$ multiplied by a constant factor, $(e-1)$, which therefore does not change the nature of the function in any way. Moreover, the discontinuity of $F(y)$ for $y=0$, will be of the same kind as that of $\varphi(y)$ in the same point.

$$F(y) \equiv (e-1)\,\varphi(y)$$

Then,

$$\lim_{\delta=0} F(\delta) = \lim_{\delta=0}\{(e-1)\varphi(\delta) - (e-1)\lim_{\delta=0}\varphi(\delta)\}.$$

if the limit of $\varphi(\delta)$ as δ approaches zero exists upon both sides of the point $y=0$. But these limits are not equal and we have

$$\lim_{\delta=0}\varphi(\delta) = A$$
$$\lim_{\delta=0}\varphi(-\delta) = B$$

and

$$\lim_{\delta=0} F(\delta) = (e-1)A$$
$$\lim_{\delta=0} F(-\delta) = (e-1)B$$

That is, when $\varphi(y)$ has a discontinuity of the first kind at the point $y=0$, $F(y)$ has a similar discontinuity at the same point. On the other hand, if

$$\lim_{\delta=0}\varphi(\delta)$$

does not exist, then

$$\lim_{\delta=0} F(\delta) \equiv (e-1)\lim_{\delta=0}\varphi(\delta)$$

can not exist, and similarly if

$$\lim_{\delta=0}\varphi(-\delta)$$

does not exist, then

$$\lim_{\delta=0} F(-\delta) \equiv (e-1)\varphi(-\delta)$$

cannot exist. That is, if $\varphi(y)$ has a discontinuity of the second kind upon either one or both sides of the point $y=0$, then $F(y)$ will have the same characteristics at that point. (For definition)

discontinuities of first and second kind see thesis of W. C. Hart p. pages 8 & 9)
Moreover if S is the jump of the function $\varphi(y)$ in the point zero we shall have, since in this case the limits must exist.

$$\left|\lim_{\delta=0} \varphi(\delta) - \lim_{\delta=0} \varphi(-\delta)\right| = |A - B| = S$$

and

$$\lim_{\delta=0} F(\delta) - \lim_{\delta=0} F(-\delta) = (e-1)|A-B| = (e-1)S$$

That is, the jump of the function $F(y)$ in the point $y=0$ is $(e-1)$ times as great as that of $\varphi(y)$ in the same point.

(But suppose that y does not have the same value in every term tho always remaining within the prescribed limits and always different from zero. In this case, we shall have a series

$$\frac{\varphi(y_1)}{1} + \frac{\varphi(y_2)}{\lfloor 2} + \frac{\varphi(y_3)}{\lfloor 3} + \cdots + \frac{\varphi(y_n)}{\lfloor n} + \cdots\cdots$$

These terms, by our hypothesis, form a convergent series, since $\varphi(y)$ always remains finite for values of y within the limits prescribed and our series

$$\frac{\varphi(y_1)}{1} + \frac{\varphi(y_2)}{\lfloor 2} + \frac{\varphi(y_3)}{\lfloor 3} + \cdots + \frac{\varphi(y_n)}{\lfloor n} + \cdots \leqq g(e-1)$$

where g is the maximum value

of $\varphi(\gamma)$ in the given interval. Moreover, when

$$1 \geq |\gamma| > 0$$

this is a series of continuous functions which is uniformly convergent. Therefore,

$$F(\gamma) \equiv \frac{\varphi(\gamma_1)}{1} + \frac{\varphi(\gamma_2)}{1^2} + \cdots\cdots + \frac{\varphi(\gamma_n)}{1^n} + \cdots\cdots$$

for

$$1 \geq |\gamma| > 0$$

is a continuous function since it is made up of the sum of an infinite number of continuous functions and the series converges uniformly.

This statement in regard to the continuity of $F(\gamma)$ is based upon the following theorems, for proof of which see thesis of W. P. Miller, 1901. pp. 30+43 also Dini, Theorie der Functionen einer veränderlichen reelen Grösse pages 140+147.

(1) A series of continuous functions represents a continuous function when it converges uniformly.

(2). A series converges uniformly when the maximum values of the individual terms form an absolutely convergent series.

Now let us take

$$y = \sin \lfloor m \pi x.$$

Then

$$F(y) \equiv F(\sin \lfloor m \pi x) \equiv f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi x)}{\lfloor n}.$$

If $\varphi(y)$ is continuous for all values of y between -1 and $+1$, zero as well as the end points included, then $f(x)$ will be continuous for all finite values of x, since in such a case we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi x)}{\lfloor n} \leqq g(e-1)$$

where g is again the maximum value of $\varphi(y)$ in the interval

$$-1 \leqq y \leqq +1,$$

for $\sin \lfloor n \pi x$ can never be greater than $+1$ nor less than -1. Then, by applying the same reasoning to this series that we used in proving that

$$F(y) \equiv \frac{\varphi(y)}{\lfloor 1} + \frac{\varphi(y_2)}{\lfloor 2} + \frac{\varphi(y_3)}{\lfloor 3} + \cdots\cdots\cdots + \frac{\varphi(y_m)}{\lfloor m} + \cdots$$

was continuous, we can easily see that $f(x)$ is continuous for all finite values of x.

But, by hypothesis, $\varphi(y)$ is discontinuous at the point $y = a$, hence we can only say that $f(x)$ is continuous

for such values of x as will satisfy the condition

$$\sin \lfloor\!\lfloor m \rfloor\!\rfloor \pi x \neq 0$$

in every term of the series; i.e. for every value of n. If x is an irrational number, then $\lfloor\!\lfloor m \rfloor\!\rfloor x$ can never be an integral number, since $\lfloor\!\lfloor m \rfloor\!\rfloor$ is an integer. $\sin \lfloor\!\lfloor m \rfloor\!\rfloor \pi x$ can only be equal to zero when $\lfloor\!\lfloor m \rfloor\!\rfloor x$ is an integral number, therefore any irrational number will fulfill the desired condition and we can say that f(x) is continuous for all irrational values of x.

So far then we have shown that if $\varphi(y)$ is a function of y which is finite and continuous for all values of y in the interval from -1 to $+1$ and is equal to zero for $y = 0$, then

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor\!\lfloor m \rfloor\!\rfloor \pi x)}{\lfloor\!\lfloor m \rfloor\!\rfloor}$$

is a continuous function for all values of x, which are not infinite. Moreover, if $\varphi(y)$ is discontinuous at the point $y = 0$, then we can only say that f(x) is continuous at the irrational points. It remains, then, for us to examine

$f(x)$ in the rational points in order to ascertain whether it is continuous or discontinuous at these points.

Every rational number can be represented as the quotient of two finite integral numbers, so the points which we are now to examine are the points where

$$x = p/q$$

where p and q are finite integral numbers and are prime to each other.

From the condition for continuity we have

$$\lim_{\delta=0} \left| f(p/q \pm \delta) - f(p/q) \right| = \sum_{n=1}^{n=\infty} \lim_{\delta=0} \left| \frac{\varphi\{\sin \lfloor n \pi (p/q \pm \delta)\} - \varphi(\sin \lfloor n \pi \, p/q)}{\lfloor n} \right|$$

must be less than σ, an arbitrarily chosen small quantity, if $f(x)$ is to be continuous in the point p/q.

(See Dini. p. 139.)

Let us now break this series up into two parts as follows;

$$\lim_{\delta=0} f(p/q \pm \delta) - f(p/q) = \sum_{n=1}^{n=\alpha-1} \lim_{\delta=0} \frac{\varphi\{\sin \lfloor n \pi (p/q \pm \delta)\} - \varphi(\sin \lfloor n \pi \, p/q)}{\lfloor n}$$

$$+ \sum_{n=\alpha}^{n=\infty} \lim_{\delta=0} \frac{\varphi\{\sin \lfloor n \pi (p/q \pm \delta)\} - \varphi(\sin \lfloor n \pi \, p/q)}{\lfloor n}$$

when α is the smallest number for-

in which k in the equation

$$p\lfloor\alpha = kq$$

is an integral number.

The first summation on the right of the sign of equality contains all those terms in which $p\lfloor m$ is not an integral multiple of q, hence $\sin \lfloor m \pi \frac{n}{q}$ is never equal to zero in any of these terms. Therefore this part of our series, being the sum of a finite number of continuous functions, forms a continuous function.

Let us now examine the second summation, namely,

$$\sum_{n=\alpha}^{\infty} \frac{1}{\delta=0} \frac{\varphi\{\sin \lfloor m \pi (\frac{n}{q}+\delta)\} - \varphi(\sin \lfloor m \pi \frac{n}{q})}{\lfloor m}.$$

This expression contains only those terms for which $p\lfloor m$ is an integral multiple of q, since by definition

$$p\lfloor\alpha = kq$$

where k is an integer. Also

$$p\lfloor\alpha+1 = (\alpha+1)kq$$

$$p\lfloor\alpha+2 = (\alpha+2)(\alpha+1)kq$$

and so on, so that

$$p\lfloor m = kq$$

and k is an integer for all values of n greater than α. Then for any term in this series

$$\sin \lfloor n \pi \, p/q \equiv \sin k q \pi \equiv 0$$

and since $\varphi(0)$ is equal to zero

$$\varphi(\sin k q \pi) \equiv 0$$

and our summation reduces to the form

$$\sum_{n=\alpha}^{\infty} \sum_{\delta=0}^{\lfloor} \frac{\varphi\{\sin \lfloor n \pi (p/q \pm \delta)\}}{\lfloor n}$$

$$= \sum_{n=\alpha}^{\infty} \sum_{\delta=0}^{\lfloor} \frac{\varphi\{\sin(\pm \lfloor n \pi \delta + k q \pi)\}}{\lfloor n}$$

$$= \sum_{n=\alpha}^{\infty} \sum_{\delta=0}^{\lfloor} \frac{\varphi(\pm \sin \lfloor n \delta \pi)}{\lfloor n}$$

In this equation the sign of $\sin \lfloor n \pi \delta$ depends upon whether k is even or odd and upon the sign of δ; i.e., when k is odd and δ is $+$, or when k is even and δ is $-$ we have $-\sin \lfloor n \pi \delta$, and when k is even and δ is $+$ or when k is odd and δ is $-$ we have $+\sin \lfloor n \pi \delta$. Three cases may arise under this summation.

(a). When the first k is even; i.e., when $p \lfloor \alpha / q$ is an even number we shall have for $+\delta$;

$$\sum_{n=\alpha}^{\infty} \frac{1}{\delta=0} \frac{\varphi(\sin \lfloor n \pi \delta)}{\lfloor n}$$

and for $-\delta$ $\sum_{n=\alpha}^{\infty} \frac{1}{\delta=0} \frac{\varphi(-\sin \lfloor n \pi \delta)}{\lfloor n}$.

(b). If k and n are odd and equal to $2k'+1$ and $2n'+1$ respectively, we have for $+\delta$,

$$\frac{1}{\delta=0} \frac{1}{(2k'+1)\, g/p} \varphi(-\sin \lfloor n \pi \delta) + \sum_{n=\alpha+1}^{\infty} \frac{1}{\delta=0} \frac{\varphi(\sin \lfloor n \pi \delta)}{\lfloor n}$$

and for $-\delta$,

$$\frac{1}{\delta=0} \frac{1}{(2k'+1)\, g/p} \varphi(\sin \lfloor n \pi \delta) + \sum_{n=\alpha+1}^{\infty} \frac{1}{\delta=0} \frac{\varphi(-\sin \lfloor n \pi \delta)}{\lfloor n}$$

There will be no more terms in $-\sin \lfloor n \pi \delta$ for $+\delta$ or $+\sin \lfloor n \pi \delta$ for $-\delta$; for, if g is contained in $p \lfloor 2n'+1$, $(2k'+1)$ times, it will be contained in $p \lfloor 2n'+2$, $(2n'+2)(2k'+1)$ times. Since $2n'+2$ is even, this whole number $(2n'+2)(2k'+1)$ is even and we have for all the rest of our terms $\varphi(+\sin \lfloor n \pi \delta)$ for $+\delta$, and $\varphi(-\sin \lfloor n \pi \delta)$ for $-\delta$.

(c) If k is odd and equal to $(2k+1)$ and n is even and equal to $2n'$, the first term will be the same as under (b). The second term will also contain $-\sin \lfloor n \pi \delta$ for $+\delta$ and $+\sin \lfloor n \pi \delta$ for $-\delta$, since $p \lfloor 2n' = (2k'+1)g$, $p \lfloor 2n'+1 = (2n'+1)(2k'+1)g$, an odd multiple of g. Thus we have two terms in $-\sin \lfloor n \pi \delta$ for $+\delta$ and $+\sin \lfloor n \pi \delta$ for $-\delta$

but no more since in the next term the even factor $(2n'+2)$ is introduced. Thus our third and last case will be for $+\delta$,

$$\frac{1}{(2k'+1)\Pi/\mu}\lim_{\delta=0}\varphi(-\sin\lfloor m\,\Pi\delta) + \frac{1}{(2k'+1)(2n'+1)\Pi/\mu}\lim_{\delta=0}\varphi(-\sin\lfloor m\,\Pi\delta)$$
$$+\sum_{n=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin\lfloor m\,\Pi\delta)}{\lfloor m}$$

and for $-\delta$,

$$\frac{1}{(2k'+1)\Pi/\mu}\lim_{\delta=0}\varphi(\sin\lfloor m\,\Pi\delta) + \frac{1}{(2k'+1)(2n'+1)\Pi/\mu}\lim_{\delta=0}\varphi(\sin\lfloor m\,\Pi\delta)$$
$$+\sum_{n=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(-\sin\lfloor m\,\Pi\delta)}{\lfloor m}$$

It may be well, before going on, to define the two kinds of discontinuity which a function may have at any point. If the limit of the function exists as the variable approaches the given value, i.e., if $\lim_{\delta=0} f(\alpha\pm\delta)$ exists and is not equal to $f(a)$, then we say that the function has a discontinuity of the first kind. $\lim_{\delta=0} f(a+\delta)$ may be equal to $f(a)$ and $\lim_{\delta=0} f(a-\delta)$ different from $f(a)$. In such a case $f(x)$ is continuous on the right and discontinuous on the left of the point a. (Or if $\lim_{\delta=0} f(a+\delta)$ is different from $f(a)$ and $\lim_{\delta=0} f(a-\delta)$ is equal to $f(a)$, then $f(x)$ is discontinuous right

and continuous right at the point x. For example see Chap. III p. 148). This kind of discontinuity is also called an ordinary discontinuity. On the other hand, if the limit in question does not exist, we say that the function has a discontinuity of the second kind. (For example see Chap. III p. 124). We can see again that this kind of a discontinuity may exist upon one or both sides of a point as was the case with the first kind, and, moreover, a function may have a discontinuity of one kind upon one side and of the other kind upon the other side. (For example see Chap. III p. 132). A further discussion of the question of these kinds of discontinuity may be found in Dini p. 52, § 31.

Now let us examine our various cases and find what kind of discontinuities $f(x)$ has for rational points.

Let us take case (a) as it is the simplest:

$$\lim_{\delta=0}\{f(\tfrac{h}{g}+\delta)-f(\tfrac{h}{g})\}=\sum_{n=\alpha}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin \lfloor n \rfloor \pi\delta)}{\lfloor n \rfloor}$$

$$= \frac{1}{\delta \doteq 0} \varphi(\sin \lfloor m \pi \delta) \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor m}$$

and

$$\frac{1}{\delta \doteq 0}\{f(\tfrac{k}{q} - \delta) - f(\tfrac{k}{q})\} = \sum_{n=\alpha}^{\infty} \frac{1}{\delta \doteq 0} \frac{\varphi(-\sin \lfloor m \pi \delta)}{\lfloor m}$$

$$= \frac{1}{\delta \doteq 0} \varphi(-\sin \lfloor m \pi \delta) \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor m}$$

It is easily seen that in this case $f(x)$ has the same characteristics in the point $x = \frac{k}{q}$ that $\varphi(y)$ has at the point $y = 0$, for our result $f(\frac{k}{q} - \delta) - f(\frac{k}{q})$ gives us merely $\varphi(\pm \sin \lfloor m \pi \delta)$ multiplied by a constant factor. Hence if $\varphi(y)$ has no limit upon either side of the point $y = 0$, $f(x)$ will have no limit upon either side of the point $x = \frac{k}{q}$, while if $\varphi(y)$ does possess such a limit $f(x)$ will also have a corresponding one, since $\sin \lfloor m \pi \delta \doteq 0$ as $\delta \doteq 0$. (For further explanation see pages 4 to 5 of this thesis).

Cases (b) and (c) may be treated under the same head.

(b)

$$\frac{1}{\delta \doteq 0}\{f(\tfrac{k}{q} + \delta) - f(\tfrac{k}{q})\} = \frac{1}{(2k+1)\frac{q}{n}} \frac{1}{\delta \doteq 0} \varphi(\pm \sin \lfloor n \pi \delta) + \sum_{n=2k+1}^{\infty} \frac{1}{\delta \doteq 0} \frac{\varphi(\pm \sin \lfloor m \pi \delta)}{\lfloor m}$$

$$\frac{1}{\delta \doteq 0}\{f(\tfrac{k}{q} - \delta) - f(\tfrac{k}{q})\} = \frac{1}{(2k+1)\frac{q}{n}} \varphi(\pm \sin \lfloor n \pi \delta) + \sum_{n=2k+1}^{\infty} \frac{1}{\delta \doteq 0} \frac{\varphi(-\pm \sin \lfloor m \pi \delta)}{\lfloor m}$$

(C) may be written

$$\lim_{\delta=0}\{f(\tfrac{k}{q}+\delta)-f(\tfrac{k}{q})\}=\left(\frac{1}{(2k+1)q^{\mu}}+\frac{1}{(2k+1)(2n+1)q^{\mu}}\right)\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)$$

$$+\sum_{n=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin\lfloor n\pi\delta)}{\lfloor n}$$

$$\lim_{\delta=0}\{f(\tfrac{k}{q}-\delta)-f(\tfrac{k}{q})\}=\left(\frac{1}{(2k+1)q^{\mu}}+\frac{1}{(2k+1)(2n+1)q^{\mu}}\right)\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)$$

$$+\sum_{n=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(-\sin\lfloor n\pi\delta)}{\lfloor n}$$

Our results will be much more evident if we write our equations as follows;

(A)

$$\lim_{\delta=0}\{f(\tfrac{k}{q}+\delta)-f(\tfrac{k}{q})\}=\frac{1}{(2k+1)q^{\mu}}\lim_{\delta=0}\varphi(-\sin\lfloor n\pi\delta)$$

$$+\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}$$

$$\lim_{\delta=0}\{f(\tfrac{k}{q}-\delta)-f(\tfrac{k}{q})\}=\frac{1}{(2k+1)q^{\mu}}\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)$$

$$+\lim_{\delta=0}\varphi(-\sin\lfloor n\pi\delta)\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}$$

(C).

$$\lim_{\delta=0}\{f(\tfrac{k}{q}+\delta)-f(\tfrac{k}{q})\}=\left(\frac{1}{(2k+1)q^{\mu}}+\frac{1}{(2k+1)(2n+1)q^{\mu}}\right)\lim_{\delta=0}\varphi(-\sin\lfloor n\pi\delta)$$

$$+\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)\sum_{n=\alpha+2}^{\infty}\frac{1}{\lfloor n}$$

$$\lim_{\delta=0}\{f(\tfrac{k}{q}-\delta)-f(\tfrac{k}{q})\}=\left(\frac{1}{(2k+1)q^{\mu}}+\frac{1}{(2k+1)(2n+1)q^{\mu}}\right)\lim_{\delta=0}\varphi(\sin\lfloor n\pi\delta)$$

$$+\lim_{\delta=0}\varphi(-\sin\lfloor n\pi\delta)\sum_{n=\alpha+2}^{\infty}\frac{1}{\lfloor n}$$

Since $\sin \underline{m}\pi\delta$ approaches zero as δ approaches zero

$$\lim_{\delta=0} \varphi(2\sin \underline{m}\pi\delta) = \lim_{\delta=0} \varphi(0+\delta)$$
$$\lim_{\delta=0} \varphi(-2\sin \underline{n}\pi\delta) = \lim_{\delta=0} \varphi(0-\delta).$$

Replacing, for the sake of brevity, the constants in our equations by β and γ we have (A) and (C) both coming under the general equations;

$$\lim_{\delta=0}\{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\} = \beta \lim_{\delta=0} \varphi(0+\delta) + \gamma \lim_{\delta=0} \varphi(0+\delta)$$

$$\lim_{\delta=0}\{f(\tfrac{h}{q}-\delta) - f(\tfrac{h}{q})\} = \beta \lim_{\delta=0} \varphi(0+\delta) + \gamma \lim_{\delta=0} \varphi(0-\delta).$$

Let us examine these equations and determine how the various kinds of discontinuity which $\varphi(y)$ may have at the point $y=0$, will affect the discontinuity of $f(x)$ in the point $x = h/q$.

(1) Let

$$\lim_{\delta=0} \varphi(0+\delta) = A, \quad \lim_{\delta=0} \varphi(0-\delta) = 0$$

i.e. $\varphi(y)$ has a discontinuity of the first kind upon <u>one</u> side of $y=0$.

Then,

$$\lim_{\delta=0}\{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\} = \gamma A$$

$$\lim_{\delta=0}\{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\} = \beta A.$$

$f(x)$ has a discontinuity of the first kind upon <u>both</u> sides of the point $x = h/q$.

(2) let

$$\lim_{\delta=0} \varphi(0+\delta) = A \quad , \lim_{\delta=0} \varphi(0-\delta) = B$$

i.e. $\varphi(y)$ has a discontinuity of the first kind upon <u>both</u> sides of the point $y = 0$.

Then,

$$\lim_{\delta=0} \{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\} = \beta B + \gamma A$$

$$\lim_{\delta=0} \{f(\tfrac{h}{q}-\delta) - f(\tfrac{h}{q})\} = \beta A + \gamma B$$

Again $f(x)$ has a discontinuity of the first kind upon <u>both</u> sides of the point $x = h/q$.

(3). let,

$$\lim_{\delta=0} \varphi(0+\delta) = A \quad , \lim_{\delta=0} \varphi(0-\delta) = ?$$

i.e. $\lim_{\delta=0} \varphi(0-\delta)$ does not exist, and $\varphi(y)$ has a discontinuity of the first kind upon <u>one</u> side of the point $y=0$, and a discontinuity of the second kind upon the other.

Then

$$\lim_{\delta=0} \{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\} = \beta ? + \gamma A.$$

$$\lim_{\delta=0} \{f(\tfrac{h}{q}-\delta) - f(\tfrac{h}{q})\} = \beta A + \gamma ?$$

$\lim_{\delta=0} \{f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})\}$ does not exist for either

$+\delta$ or $-\delta$ and hence $f(x)$ has a discontinuity of the second kind upon both sides of the point $x = b_1$.

(4). Let,

$$\lim_{\delta \doteq 0} \varphi(0+\delta) = ?\,, \quad \lim_{\delta \doteq 0} \varphi(0-\delta) = 0$$

i.e. $\varphi(y)$ has a discontinuity of the second kind upon one side of the point $y = 0$ and is continuous upon the other side. Then,

$$\lim_{\delta \doteq 0} \{f(b_2+\delta) - f(b_2)\} = ?$$

$$\lim_{\delta \doteq 0} \{f(b_2-\delta) - f(b_2)\} = ?$$

Again $f(x)$ has a discontinuity of the second kind upon both sides of the point $y = 0$.

(5). Let,

$$\lim_{\delta \doteq 0} \varphi(0+\delta) = ?\,, \quad \lim_{\delta \doteq 0} \varphi(0-\delta) = ?$$

i.e. $\varphi(y)$ has a discontinuity of the second kind upon both sides of the point $y = 0$. Then,

$$\lim_{\delta \doteq 0} \{f(b_1+\delta) - f(b_1)\} = ? + ?$$

$$\lim_{\delta \doteq 0} \{f(b_1-\delta) - f(b_1)\} = ? + ?$$

$f(x)$ has a discontinuity of the second kind upon both sides of the point $x = b_1$.

I'm summing up our results we may say;

If $\varphi(y)$ is a function of y, which, for all values of y between -1 and $+1$, the end points included, is finite and determined and moreover, is continuous except at the point $y=0$, and $\varphi(y)=0$ for $y=0$, then

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\varphi(2n\pi i \pm \pi x)}{\lfloor n}$$

will have the following characteristics $f(x)$ will be continuous at every irrational point. For every rational point for which k in the expression $p/q = k/q$ is even $f(x)$ will have the same kind of discontinuity that $\varphi(y)$ has at the point $y=0$. For the rational points for which k is odd $f(x)$ behaves thus;

(1). If $\varphi(y)$ has a discontinuity of the first kind upon both sides of the point $y=0$, or only upon one side and is continuous upon the other, then $f(x)$ has a discontinuity of the first kind upon both sides of the given point.

(2). If $\varphi(y)$ has a discontinuity of the second kind upon both sides of the point

$\psi = 0$, or only when one side and one of the first kind or is continuous when the other side, then $f(x)$ has a discontinuity of the second kind when both sides of the point in question.

Moreover, since

$$\lim_{\delta=0}(f(\tfrac{h}{q}+\delta) - f(\tfrac{h}{q})) = \sum_{n=\alpha}^{\infty} \lim_{\delta=1} \frac{\varphi(\sin \lfloor n \pi \delta}{\lfloor n} = \lim_{\delta=0} \frac{\varphi(\sin \lfloor n \pi \delta)}{1} \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$$

as we increase q, α must increase and therefore $\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$ must decrease. If S is the jump of the function $\varphi(\psi)$ in the point $\psi = 0$, $\left(\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}\right) S$ will be the jump of the function $f(x)$ in the point h/q. Since $\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$ decreases as q increases, there can be only a finite number of q's which will render $\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$ greater than a definite, fixed number β. If we take $\beta S = \tau$, and τ is also a definite, fixed value, then there can only be a finite number of q's which will render the inequality

$$\left(\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}\right) S > \tau$$

hold true. i. e. there will be only a finite number of points in which the discontinuity of $f(x)$ is greater than some definite fixed quantity.

Example.

Given, $\varphi(y) = \lim_{m=\infty} (y^2)^{1/m}$. Hence $\varphi(y)$ is equal to 1 for y different from zero, and equal to 0 for y equal to zero. Therefore it has a discontinuity of the first kind upon both sides of the point $y=0$, and is continuous at every other point. This function then fulfills all the conditions imposed upon the general function $\varphi(y)$ in the demonstration and we may use it as an illustration.

By definition (page 8)

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\lim_{m=\infty}} \frac{(\sin \lfloor n \pi x)^{2/m}}{\lfloor n}$$

Since $\lim_{m=\infty} (y^2)^{1/m} = 1$ for all values of y except $y=0$, $f(x)$ will be equal to a uniformly convergent series which is always equal to $(e-1)$ for all irrational values of x. i.e.

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\lim_{m=\infty}} \frac{(\sin \lfloor n \pi x)^{2/m}}{\lfloor n} = \sum_{n=1}^{\infty} \frac{1}{\lfloor n} = (e-1)$$

This follows from the fact that $\sin \lfloor n \pi x \neq 0$ for all irrational values of x (see page 9). $\varphi(y)$ is continuous for all values of y different from 0. Hence for irrational values of x, $f(x)$ is made up of a uniformly convergent series of con-

tinuous functions and is, therefore a continuous function for these points.

For the rational points we must apply our ordinary test for discontinuity, namely,

$$\left| f\left(\frac{h}{g} \pm \delta\right) - f\left(\frac{h}{g}\right) \right| = \left| \sum_{n=1}^{n=\infty} \frac{1}{\lfloor n} \left(\sin^2 \lfloor n \left(\frac{h}{g} \pm \delta\right)\pi\right)^{\frac{2}{n}} - \left(\sin^2 \lfloor n \frac{h}{g} \pi\right)^{\frac{2}{n}} \right|$$

must be less than some arbitrarily chosen small positive number σ for this δ and every δ smaller than this.

Every term in this series for which $\lfloor n$ is not a multiple of q will vanish since in those terms our numerator becomes simply $1-1$. Moreover for those terms in which $\lfloor n$ is a multiple of q, $\left(\sin^2 \lfloor n \pi \frac{h}{g}\right)^{\frac{2}{n}} = 0$. Our series then reduces to

$$\left| f\left(\frac{h}{g} \pm \delta\right) - f\left(\frac{h}{g}\right) \right| \equiv \left| \sum_{n=\alpha}^{n=\infty} \frac{1}{\lfloor n} \sum_{m=0}^{m=\infty} \frac{\left\{\sin^2(K\pi + \lfloor m \pi \delta)\right\}^{\frac{2}{m}}}{\lfloor m} \right|$$

$$\equiv \left| \sum_{n=\alpha}^{n=\infty} \frac{1}{\lfloor n} \sum_{m=0}^{m=\infty} \frac{(\pm \sin \lfloor m \pi \delta)^{\frac{2}{m}}}{\lfloor m} \right| \quad \text{(see page 12)}$$

$$\equiv \left| \sum_{n=\alpha}^{n=\infty} \frac{1}{\lfloor n} \right|$$

This last expression follows from the fact that $(\pm \sin \lfloor m \pi \delta)^{\frac{2}{m}} \neq 0$ and hence $(\sin \lfloor m \pi \delta)^{\frac{2}{m}}$ as m approaches infinity

has the limit 1. Since $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ is a fixed finite number, it cannot be made less than our arbitrarily chosen σ, then $f(x)$ is discontinuous at the rational points.

Moreover, since the limit of $\{(\pm \sin k\pi\delta)^2\}^{\frac{1}{m}}$ as $m = \infty$ & $\delta = 0$ exists and is equal to 1, which is different from 0, the value of the expression when $\delta = 0$, this discontinuity is of the first kind. The discontinuity is independent of the sign of δ and of the nature of k, whether even or odd since $(-\sin k\pi\delta)^2 = (\sin k\pi\delta)^2$ hence the discontinuity will be of the same kind on both sides of the point $x = \frac{p}{q}$.

In order that the behavior of this function may be the more easily understood the following table of values and plat of certain points on the next page, is given. It may be seen by reference to the plat that any irrational point may be approached thro a series of rational points which grow dense around each irrational point and in which the function approaches nearer

and nearer to its value at that point. This conception will make it much easier to see how the function can be continuous for every rational point and discontinuous for every rational point.

Tabulated values of x and $f(x)$.

x	$f(x)$
0	0.00
1	0.00
1/2	1.00
1/3	1.50
2/3	1.50
1/4	1.66
3/4	1.66
1/5	1.70
2/5	1.70
3/5	1.70
4/5	1.70
1/6	1.50
5/6	1.50
1/8	1.66
3/8	1.66
5/8	1.66

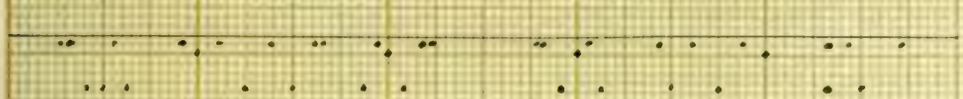
x	$f(x)$
7/8	1.66
1/9	1.71
2/9	1.71
4/9	1.71
5/9	1.71
7/9	1.71
8/9	1.71
1/10	1.66
3/10	1.66
7/10	1.66
9/10	1.66
1/12	1.66
5/12	1.66
7/12	1.66
11/12	1.66
1/16	1.71

x	$f(x)$
3/16	1.71
5/16	1.71
7/16	1.71
9/16	1.71
11/16	1.71
13/16	1.71
15/16	1.71
1/18	1.71
5/18	1.71
7/18	1.71
11/18	1.71
13/18	1.71
17/18	1.71

$f(x)$.

Graph of the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{m=\infty} \left(\frac{(\sin m\pi x)^{2/m}}{\lfloor m} \right)$$



(e-1)

1.5

1.

0

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X

Scale - 10 Squares = 1 unit.

Section II.

In this section we shall follow out the method of condensation devised by Hankel. A discussion of this method may be found in Dini, Theorie der Functionen einer veränderlichen reellen Grösse pages 157-163.

Let our $\varphi(y)$ now be taken as a function which fulfills the conditions laid upon the function $\varphi(y)$ used in section I. Then we have $\varphi(y)$ a function of y, which, within the limits -1 and $+1$, the end points included, is finite and determined and is continuous except at the point $y=0$, and for $y=0$, $\varphi(y)=0$.

Our problem is the same as that in section I, namely to build up from this function $\varphi(y)$, which is discontinuous at one point, a new function $f(x)$, which shall be discontinuous at a set of points every where dense within any given interval, say for all rational points.

We shall now define our $F(y)$ as follows;

$$F(y) \equiv \sum_{n=1}^{\infty} \frac{\varphi(y)}{n^s} \quad ; \quad s > 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is a convergent series when $s > 1$ we may write our equation

$$F(y) \equiv \varphi(y) \sum_{n=1}^{\infty} \frac{1}{n^s} \equiv A.\varphi(y)$$

We may now apply the same reasoning to $F(y)$ that we did to the similar function on pages 4 to 7, section I, since we have the same form of $F(y)$ as before, the only difference being the substitution of A as the constant multiplier instead of $(c-1)$.

$F(y)$ then will have the same kind of discontinuity at the point $y=0$ that $\varphi(y)$ has at that point, and the jump of $F(y)$ in the point $y=0$ is A times as great as that of $\varphi(y)$ in the same point. In case y has different values in the different terms i.e.

$$F(y) \equiv \frac{\varphi(y_1)}{1} + \frac{\varphi(y_2)}{2^s} + \frac{\varphi(y_3)}{3^s} + \cdots + \frac{\varphi(y_n)}{n^s} + \cdots$$

$F(y)$ will be continuous for all values of $y_1, y_2, y_3 \cdots$ different from zero, and if $\varphi(y)$ were continuous at $y=0$ then $F(y)$ would be continuous at all points from -1 to $+1$.

– Let us now make the substitution

$$y = \sin n\pi x.$$

Then

$$F(y) \equiv F(\sin n\pi x) \equiv f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin n\pi x)}{n^s}$$

By the same reasoning which we used in the similar case given in section I, pages 8 and 9, we see that $f(x)$ is continuous at all irrational points. It remains then, for us to examine the continuity or discontinuity of the function at the rational points, namely when $x = h/k$.

At these points we have;

$$\lim_{\delta=0}\{f(h/k \pm \delta) - f(h/k)\} = \sum_{n=1}^{\infty} \lim_{\delta=0} \frac{\varphi\{\sin n\pi(h/k \pm \delta)\} - \varphi(\sin n\pi h/k)}{n^s}$$

Or, separating our series;

$$\lim_{\delta=0}\{f(h/k \pm \delta) - f(h/k)\} \equiv \sum \lim_{\delta=0} \frac{\varphi\{\sin n\pi(h/k \pm \delta)\} - \varphi(\sin n\pi h/k)}{n^s}$$

$$hn \neq Kg$$

$$+ \sum \lim_{\delta=0} \frac{\varphi\{\sin n\pi(h/k \pm \delta)\} - \varphi(\sin n\pi h/k)}{n^s}$$

$$hn = Kg$$

In this expression, the first summation on the right side of the sign of equality contains all those terms in which hn is not an integral multi-

difference of δ. The second summation is composed of all those terms for which pn is a multiple of q. Since $\sin n\pi\frac{p}{q}$ is not zero in any of the terms of the first summation, $\varphi(\sin n\pi\frac{p}{q})$ is a continuous function in each of these terms. (By the same reasoning as that employed in regard to the irrational points) we see that $\sum_{n=1}^{\infty}\frac{\varphi(\sin n\pi\frac{p}{q})}{n^s}$ $pn \neq kq$ represents a continuous function and therefore

$$\sum_{n=1}^{\infty}\Big|_{\delta=0}\frac{\varphi(\sin n\pi(\frac{p}{q}\pm\delta)-\varphi(\sin n\pi\frac{p}{q})}{n^s}\Big|_{pn\neq kq}=0.$$

We may now write our expression which we were examining as

$$\Big|_{\delta=0}\{f(\tfrac{p}{q}\pm\delta)-f(\tfrac{p}{q})\}=\sum_{n=1}^{\infty}\Big|_{\delta=0}\left[\frac{\varphi(\sin n\pi(\frac{p}{q}\pm\delta)-\varphi(\sin n\pi\frac{p}{q})}{n^s}\right]_{pn=kq}$$

Since $pn=kq$ we may write this

$$\Big|_{\delta=0}\{f(\tfrac{p}{q}\pm\delta)-f(\tfrac{p}{q})\}=\sum_{n=1}^{\infty}\Big|_{\delta=0}\left[\frac{\varphi(\sin(k\pi\pm n\pi\delta)-\varphi(\sin k\pi)}{n^s}\right]_{pn=kq}$$

$$=\sum_{n=1}^{\infty}\Big|_{\delta=0}\left[\frac{\varphi(\pm\sin n\pi\delta)-\varphi(0)}{n^s}\right]_{pn=kq}$$

$$=\sum_{n=1}^{\infty}\Big|_{\delta=0}\left[\frac{\varphi(\pm\sin n\pi\delta)}{n^s}\right]_{pn=kq}.$$

We are enabled to drop the $\varphi(\sin k\pi)$ since it is equal to $\varphi(0)$, which is zero by our hypothesis. The sign of $\sin n\pi\delta$

depends upon the sign of δ and upon whether k in that particular term is even or odd. (For similar case see page 12). In those terms in which k is even the sign of $\sin n\pi\delta$ will be the same as that of δ and in those terms in which k is odd $\sin n\pi\delta$ will have a + sign for $-\delta$ and a − sign for $+\delta$.

Let us now separate our series into two parts, one containing the terms in which k is even and equal to $2k'$ and the other containing the terms in which k is odd and equal to $2k'+1$. We have then

$$\lim_{\delta=0}\left\{f\left(\tfrac{h}{q}+\delta\right)-f\left(\tfrac{h}{q}\right)\right\}=\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\psi(\sin n\pi\delta)}{n^s}$$

$$n=2k'q$$

$$+\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\psi(-\sin n\pi\delta)}{n^s}$$

$$n=(2k'+1)q$$

$$\lim_{\delta=0}\left\{f\left(\tfrac{h}{q}-\delta\right)-f\left(\tfrac{h}{q}\right)\right\}=\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\psi(-\sin n\pi\delta)}{n^s}$$

$$n=2k'q$$

$$+\sum\lim_{\delta=0}\frac{\psi(\sin n\pi\delta)}{n^s}$$

$$n=(2k'+1)q.$$

Now let us see under what circumstances k will be odd and under what circumstances it will be even. To determine the value of k

we have the following equation, viz;

$$K = \frac{p^n \pm 1}{q}$$

Since p and q are prime to each other, they both may be odd, or one may be odd and the other even; but they cannot both be even, since then they would have the common factor 2, and hence would not be prime to each other. Let us now examine these different cases.

(a) Let p be odd and q even. Then the number p^n will be alternately odd and even according as n is odd or even and K will vary for different values of n.

(b) Let p and q both be odd. Here we may use the same reasoning and arrive at the same result as in (a).

(c) Let p be even and q odd. Then p^n must always be an even number and q will be contained in p^n an even number of times since an odd number cannot be contained in an even number an odd number of times. Hence K will always be even.

We see, then, from our reasoning

in these three cases that when p is even, k is always even and when p is odd, k is sometimes odd and sometimes even. Our series then may be written as follows;

For (a) and (b),

$$\lim_{\delta=0}\left\{f\left(\tfrac{h}{q}+\delta\right)-f\left(\tfrac{h}{q}\right)\right\}=\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin n\pi\delta)}{n^s}$$

$$p n=2k'q$$

$$+\sum\lim_{\delta=0}\frac{\varphi(-\sin n\pi\delta)}{n^s}$$

$$p n=(2k'+1)q.$$

$$=\lim_{\delta=0}\varphi(\sin n\pi\delta)\sum_{n=1}^{\infty}\frac{1}{n^s}\qquad p n=2k'q$$

$$+\lim_{\delta=0}\varphi(-\sin n\pi\delta)\sum_{n=1}^{\infty}\frac{1}{n^s}\qquad p n=(2k'+1)q.$$

$$\lim_{\delta=0}\left\{f\left(\tfrac{h}{q}-\delta\right)-f\left(\tfrac{h}{q}\right)\right\}=\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\varphi(-\sin n\pi\delta)}{n^s}$$

$$p n=2k'q$$

$$+\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin n\pi\delta)}{n^s}$$

$$p n=(2k'+1)q.$$

$$=\lim_{\delta=0}\varphi(-\sin n\pi\delta)\sum\frac{1}{n^s}\qquad p n=2k'q$$

$$+\lim_{\delta=0}\varphi(\sin n\pi\delta)\sum\frac{1}{n^s}\qquad p n=(2k'+1)q.$$

For (c)

$$\lim_{\delta=0}\left\{f\left(\tfrac{h}{q}+\delta\right)-f\left(\tfrac{h}{q}\right)\right\}=\sum_{n=1}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin n\pi\delta)}{n^s}$$

$$p n=2k'q$$

$$= \lim_{\delta=0} \varphi(\sin n\pi\delta) \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad p n = 2k'q.$$

$$\lim_{\delta=0}\{f(\tfrac{p}{q}-\delta) - f(\tfrac{p}{q})\} = \sum_{n=1}^{\infty} \lim_{\delta=0} \frac{\varphi(-\sin n\pi\delta)}{n^s} \qquad p n = 2k'q.$$

$$= \lim_{\delta=0} \varphi(-\sin n\pi\delta) \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad p n = 2k'q.$$

By following out the same line of reasoning as that employed in the similar case on page 16, we find that in those points, $x = \frac{p}{q}$, where p is even $f(x)$ has the same characteristics that $\varphi(y)$ has at the point $y = 0$. This may be made still more plain by reference to pages 4 and 5 also. In the same manner, by reference to page 16 and those following it, we find that if $\varphi(y)$ has a discontinuity of the first kind upon <u>both</u> sides of the point $y = 0$, or if it is continuous upon <u>one</u> side and has such a discontinuity upon <u>the other</u>, then $f(x)$ has a discontinuity of the first kind upon <u>both</u> sides of the point $x = \frac{p}{q}$, when p is <u>odd</u>. Furthermore, if $\varphi(y)$ has a discontinuity of the second kind upon <u>either one</u>, or upon <u>both</u>

sides of the point $y=1$, then $f(x)$ has a discontinuity of the second kind upon both sides of the point $x=\frac{p}{q}$, when p is odd.

Summing up the results attained in this section we have,-

Given, $\varphi(y)$, a function of y, which, for all values of y between -1 and $+1$, the end points included, is finite and determined and is continuous within this interval except at the point $y=1$, and $\varphi(y)=0$ for $y=1$. Then,

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\varphi(\sin n\pi x)}{n^s} \qquad ; s>1$$

is a function of x which is finite determined and continuous for all irrational values of x, and is finite and determined for rational values. However, in the rational points, $x=\frac{p}{q}$, in which p is even $f(x)$ has the same characteristics, so far as discontinuity is concerned, that $\varphi(y)$ has in the point $y=0$. In the rational points in which p is odd, if $\varphi(y)$ has an ordinary discontinuity on one side of the point $y=0$, and is continuous upon the other,

or if it has such a discontinuity upon both sides of the point $y=a$, then $f(x)$ has a discontinuity of the first kind upon both sides of the point $x=\frac{h}{q}$. If $\varphi(y)$ has a discontinuity of the second kind upon either side or upon both sides of the point $y=a$, then $f(x)$ has such a discontinuity upon both sides of the point $x=\frac{h}{q}$, when p is odd.

Moreover, by the same reasoning as in section I. page we see that $f(x)$ can have only a finite number of discontinuities which are greater than any fixed finite value T, since there will only be a finite number of q's which will satisfy the equation $(Rm\ S>\sigma$ as in the similar case. (Page 22).

Example.

Given

$$\varphi(y) = \lim_{m=\infty} \{1 - (\cos y)^{2m}\}$$ (See page 158.)

Hence $\varphi(y) = 1$ for $y \neq 0$ and $\varphi(y) = 0$ for $y = 0$; i.e. $\varphi(y)$ has a discontinuity of the first kind upon both sides of the point $y = 0$ and is continuous at all other points within the limits

$$-1 - \gamma \leq y \leq 1.$$

We have then, letting $s = 2$;

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\lim_{m=\infty} \{1 - [\cos(\sin n\pi x)]^{2m}\}}{n^2}$$

Since $\varphi(y)$ is continuous for $y \neq 0$, and $\sin n\pi x \neq 0$ for irrational values of x, $f(x)$ will be continuous for irrational values of x for the maximum value of $f(x)$ in any interval is 1 and

$$* \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

an absolutely convergent series (See page 75)

Let us now apply our ordinary test for discontinuity at the rational points. For these points we have, then

$$\left| f(h/g \pm \delta) - f(h/g) \right| = \left| \sum_{n=1}^{\infty} \frac{\lim_{m=\infty} \left[1 - \{\cos \sin \pi (h/g \pm \delta) n)\}^{2m} + 1 + \{\cos(\sin \pi h/g n)\}^{2m} \right]}{n^2} \right|$$

and this summation must be less than some arbitrarily chosen small positive

* See Chystal page 343.

number $\frac{p}{q}$ of $f(x)$ is continuous at the point $x = \frac{p}{q}$. Since since $\varphi(y)$ is continuous for $y \neq 0$, all those terms in which

$$p \, n \neq K q$$

will drop out since they will represent continuous functions. We shall have then

$$\left| f\left(\frac{p}{q} \pm \delta\right) - f\left(\frac{p}{q}\right) \right| = \left| \sum_{K=1}^{\infty} \frac{1}{n^2} \lim_{m=\infty} \left[1 - \{\cos(\sin K\pi \pm n\delta\pi)\}^{2m} - 1 + \{\cos(\sin K\pi)\}^{2m} \right] \right|$$

where

$$\frac{p}{q} n = K$$

and hence

$$n = \frac{qK}{p}$$

Since

$$\sin K\pi = 0$$

we have at the last part of the terms of our summation

$$-1 + \{\cos 0\}^{2m} = 0$$

and our summation becomes

$$\left| f\left(\frac{p}{q} \pm \delta\right) - f\left(\frac{p}{q}\right) \right| = \left| \sum_{K=1}^{\infty} \frac{1}{n^2} \lim_{m=\infty} \left[1 - \{\cos(\pm \sin n\pi\delta)\}^{2m} \right] \right|_{n = \frac{qK}{p}}$$

$$= \sum_{K=1}^{\infty} \frac{1}{n^2} \quad : n = K\frac{q}{p}$$

This is a fixed quantity and cannot be made less than σ for any given number $\frac{p}{q}$, hence $f(x)$ has a discontinuity of the

first kind at every rational point.

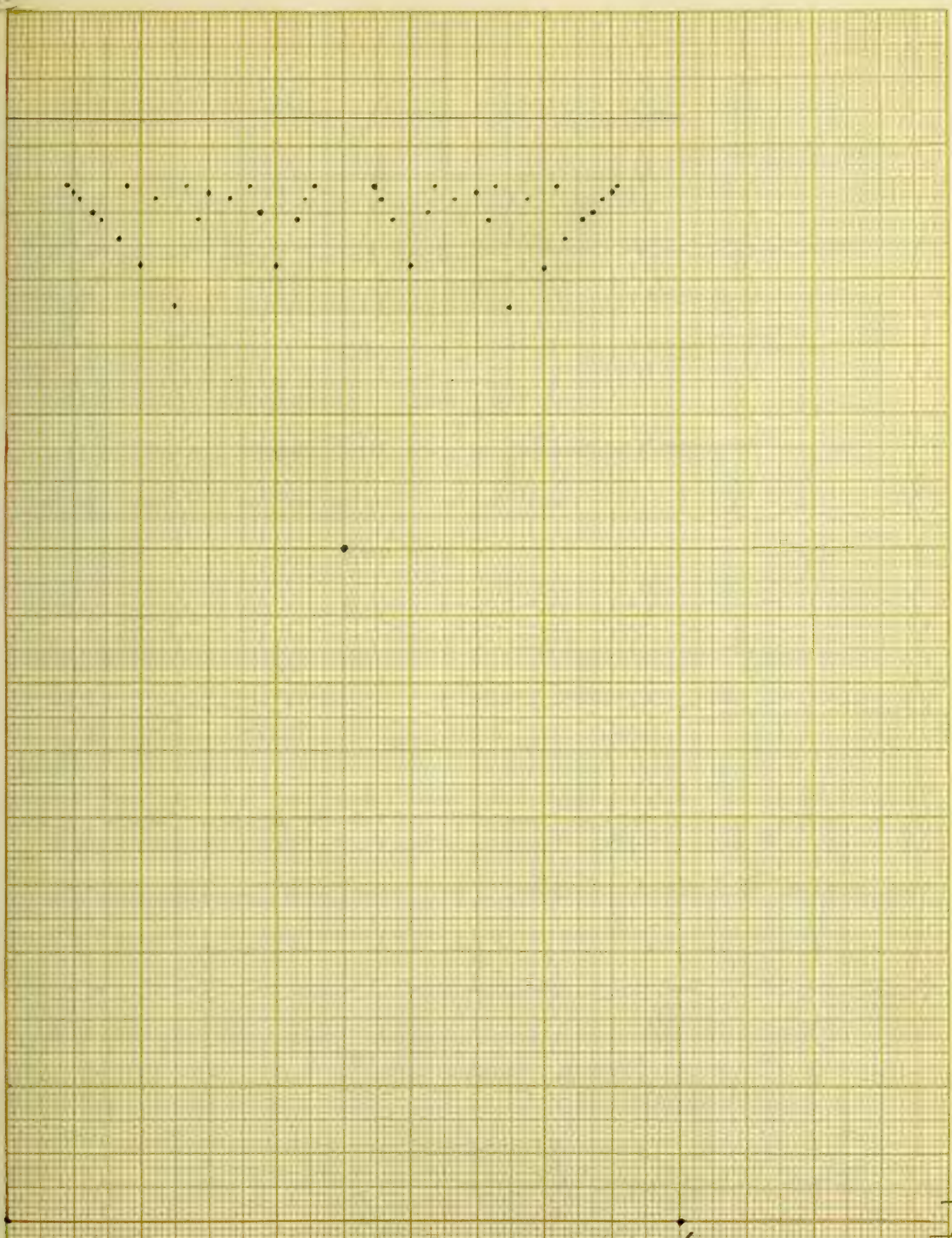
As in the problem given at the end of section I, a table of values and a plot of some of the values of $f(x)$ and x is given in order to make the nature of the function more clear. It will be seen from the plotted values of $f(x)$ that it behaves in very much the same manner that the function plotted at the end of section I behaved.

Table of Values of x and $f(x)$.

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0	0	$1/6$	1.46+	$7/8$	1.5-+	$1/11$	1.5-4-
1	0	$5/6$	1.46+	$1/9$	1.5-2-	$2/11$	1.5-4-
$1/2$	1	$1/7$	1.49+	$2/9$	1.5-2-	$3/11$	1.5-4-
$1/3$	1.25-	$2/7$	1.49+	$4/9$	1.5-2-	$4/11$	1.5-4-
$2/3$	1.25-	$3/7$	1.49+	$5/9$	1.5-2-	$5/11$	1.5-4-
$1/4$	1.36+	$4/7$	1.49+	$7/9$	1.5-2-	$6/11$	1.5-4-
$3/4$	1.36+	$5/7$	1.49+	$8/9$	1.5-2-	$7/11$	1.5-4-
$1/5$	1.42+	$6/7$	1.49+	$1/10$	1.5-3-	$8/11$	1.5-4-
$2/5$	1.42+	$1/8$	1.5-+	$3/10$	1.5-3-	$9/11$	1.5-4-
$3/5$	1.42+	$3/8$	1.5-+	$7/10$	1.5-3-	$10/11$	1.5-4-
$4/5$	1.42+	$5/8$	1.5-+	$9/10$	1.5-3-		

$f(x) = 1.64+$ for irrational x.

f(x)



Graph of the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\lim_{m=\infty} [1 - \{\cos(\sin n\pi x)\}^{2m}]}{n^2}$$

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Scale - 10 Squares = 1 unit.

Section III

In this section we shall treat our function according to the method devised by Cantor. For a discussion of this method see Dini, Theorie der Functionen einer veränderlichen reelen Grösse, page 188.

Let $\varphi(y)$ be a function of y, which, for all values of y between the limits a and b, is finite and determined, and moreover, is continuous within this interval, except at the point $y=0$.

Our problem now is to build up from this function $\varphi(y)$ a new function of x, $f(x)$, which shall be discontinuous at <u>any set of points which we may choose</u>. These points may be everywhere dense, or distributed at regular finite intervals, or at irregular intervals, the only restriction placed upon them being that they shall form an abzählbar Menge.

Let us choose any a set of positive, real numbers;

$$c_1, c_2, c_3, \cdots, c_n, \cdots$$

such that they form a convergent series i.e., they have a finite sum which is equal to k.

Again let us choose another set of numbers

$$w_1, w_2, w_3, \cdots w_n \cdots$$

which are the abscissae of $f(x)$ at the points at which we wish $f(x)$ to be discontinuous.

We are now ready to form our series as follows;

$$f(x) \equiv \sum_{n=1}^{\infty} c_n \varphi(x - w_n) \; ; \quad a < |x - w_n| < b.$$

This summation will represent a continuous function for all values of x which are not equal to any of our w's, for we can write it thus;

$$f(x) \equiv \sum_{n=1}^{\infty} c_n \varphi(x - w_n) < g \sum_{n=1}^{\infty} c_n \; ; a < |x - w_n| < b.$$

where g is the maximum value of $\varphi(y)$ in the interval from a to b; i.e. the series

$$\sum_{n=1}^{\infty} c_n \varphi(x - w_n) \; ; \quad a < |x - w_n| < b$$

converges uniformly (see page 7) and therefore, since $\varphi(y)$ is continuous for all values of y different from zero, within the limits a and b, $f(x)$ is continuous between the limits $a < |x - w_n| < b$

for all values of x which are not equal to any of the w^2, since in that case $x - w_n \neq 1$ for every term (see page 42).

For the values of x for which $x = w_k$, we may write our function;

$$f(x) = \sum_{n=1}^{k-1} C_n \varphi(x - w_n) + \sum_{n=k+1}^{\infty} C_n \varphi(x - w_n) + C_k \varphi(x - w_k).$$

We can see from our reasoning at the foot of the preceding page and the top of this one that the first two terms on the right of the sign of equality form a continuous function, but the last term, namely,

$$C_k \varphi(x - w_k)$$

is discontinuous at the point $x = w_k$ since $\varphi(y)$ is by definition discontinuous at the point $y = 0$. Therefore, for all points, for which x is equal to any of the numbers

$$w_1, w_2, w_3, \ldots w_n \ldots$$

$f(x)$ is discontinuous. Moreover the discontinuity of $f(x)$ in these points is of the same nature as that of $\varphi(y)$ in the point $y = 0$ since it depends upon the discontinuity of $C_k \varphi(x - w_k)$.

If S represent the jump of

the function $\varphi(\gamma)$ in the point $\gamma = a$, then $c_k S$ will represent the jump of $f(x)$ in the point $x = w_k$. On account of the convergence of the series

$$c_1 \; c_2 \; c_3 \cdots c_n \cdots$$

there can only be a finite number of c's which will satisfy the condition

$$c_k S > |\sigma|$$

where σ is some definite fixed quantity, hence there can only be a finite number of points in which the amount of the discontinuity of $f(x)$ is greater than some fixed quantity σ.

In summing up the results obtained by this method we may say:-

If $\varphi(\gamma)$ is a function of γ, which, for any interval, $a < \gamma < b$, is finite and determined, and is continuous for all values within the interval except at the point $\gamma = 0$, then

$$f(x) \equiv \sum_{n=1}^{\infty} c_n \varphi(x - w_n)$$

where $w_1, w_2, \cdots w_n \cdots$ is any arbitrary merge of points and

$$c_1, c_2, c_3 \cdots c_n \cdots$$

form a convergent series, will have

the same kind of discontinuity at the points $u_1, u_2, u_3 \cdots\cdots u_n \cdots\cdots$ that $\varphi(y)$ has at the point $y=0$, and will be continuous at every other point. Furthermore, $f(x)$ cannot be totally discontinuous since the algebraic numbers do not form an abzählbar menge and we have based our proof upon the hypothesis that

$$u_1, u_2, u_3 \cdots\cdots u_n \cdots$$

formed an abzählbar menge. $f(x)$ may however be discontinuous at every rational point since it has been shown by Cantor that the rational numbers form an abzählbar menge.

Example.

Given

$$\varphi(y) \equiv \frac{1}{m=\infty} (y)^{\frac{2}{m}}.$$

This is identically the same function as that used at the end of section 1 and we have again $\varphi(y) = 1$ for $y \neq 0$, and $\varphi(y) = 0$ for $y = 0$. Let

$$C_n = \frac{1}{n}.$$

We have then

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{m=\infty} \frac{(x-u_n)^{\frac{2}{m}}}{n}$$

Let us choose our menge of w's as the + rational numbers and arrange them as follows

$$0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \ldots$$

Since no irrational number belongs to this menge $(X\ w_n)$ cannot be equal to zero ~~for~~ any irrational number and by reasoning similar to that used on page 43 we see that

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad (e-1)$$

and is continuous for all irrational values of x.

For the rational number we shall have

$$f(x) = \frac{1}{1}(x-0)^{\frac{2}{m}} + \frac{1}{12}(x-1)^{\frac{2}{m}} + \frac{1}{13}(x-\tfrac{1}{2})^{\frac{2}{m}} - + \ldots$$

It will easily be seen that one of these terms will become zero for every rational point and since 1 is the jump of $\psi(y)$ in the point $y=0$, $\frac{1}{m}$ will be the jump of $f(x)$ in the given point, m being the number of the term which drops out. On the next page will be found a table of values of x and the corresponding values of $f(x)$ and fol-

leaving that a plot of these values.

Table of Values of x and the Corresponding Values of $f(x)$.

x	$f(x)$
0	$.72 \cdot (e-2)$
1	$1.22 = (e-1) - \frac{1}{2}$
$\frac{1}{2}$	$1.55+ = (e-1) - \frac{1}{6}$
$\frac{1}{3}$	$1.677+ = (e-1)\ \frac{1}{24}$
$\frac{2}{3}$	$1.71+ = (e-1) - \frac{1}{120}$
$\frac{1}{4}$	$1.715+ = (e-1) - \frac{1}{720}$

For all irrational values of x we have

$$f(x) = (e-1) = 1.72-$$

(For plot of this function see next page).

$f(x)$.

Graph of the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{m=\infty} \frac{(x-u_{(m)})^{2/m)}}{km} \quad ; u_k = \frac{n}{g}$$

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O.

X

Scale - 10 Squares = 1 Unit.

On page 42 the statement is made that by means of the method set forth in this section $f(x)$ may be made discontinuous at a set of points at finite distances from each other, either at regular or irregular intervals. In order to make this more clear the following examples are given.

Let us take the same function of y as that used on page 46, namely

$$f(y) = \frac{1}{\lfloor m} \overset{m=\infty}{=} (y)^{2/m}.$$

We shall now build up our function $f(x)$, making it discontinuous at every integral value of x. Our menge of w's then becomes the menge of integers, namely;

$$1, 2, 3, 4, \text{- - - - - - - - - - - - - - - -}$$

and

$$w_k = k$$

As in the former example we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{\lfloor m} \overset{m=\infty}{(x - w_n)^{2/m}} \quad ; c_n \equiv \frac{1}{\lfloor m}$$

$f(x)$ then will be continuous for all values of x which are not integral but will be discontinuous for every integral

value. The nature of the function may be easily seen from the following table of values and plot.

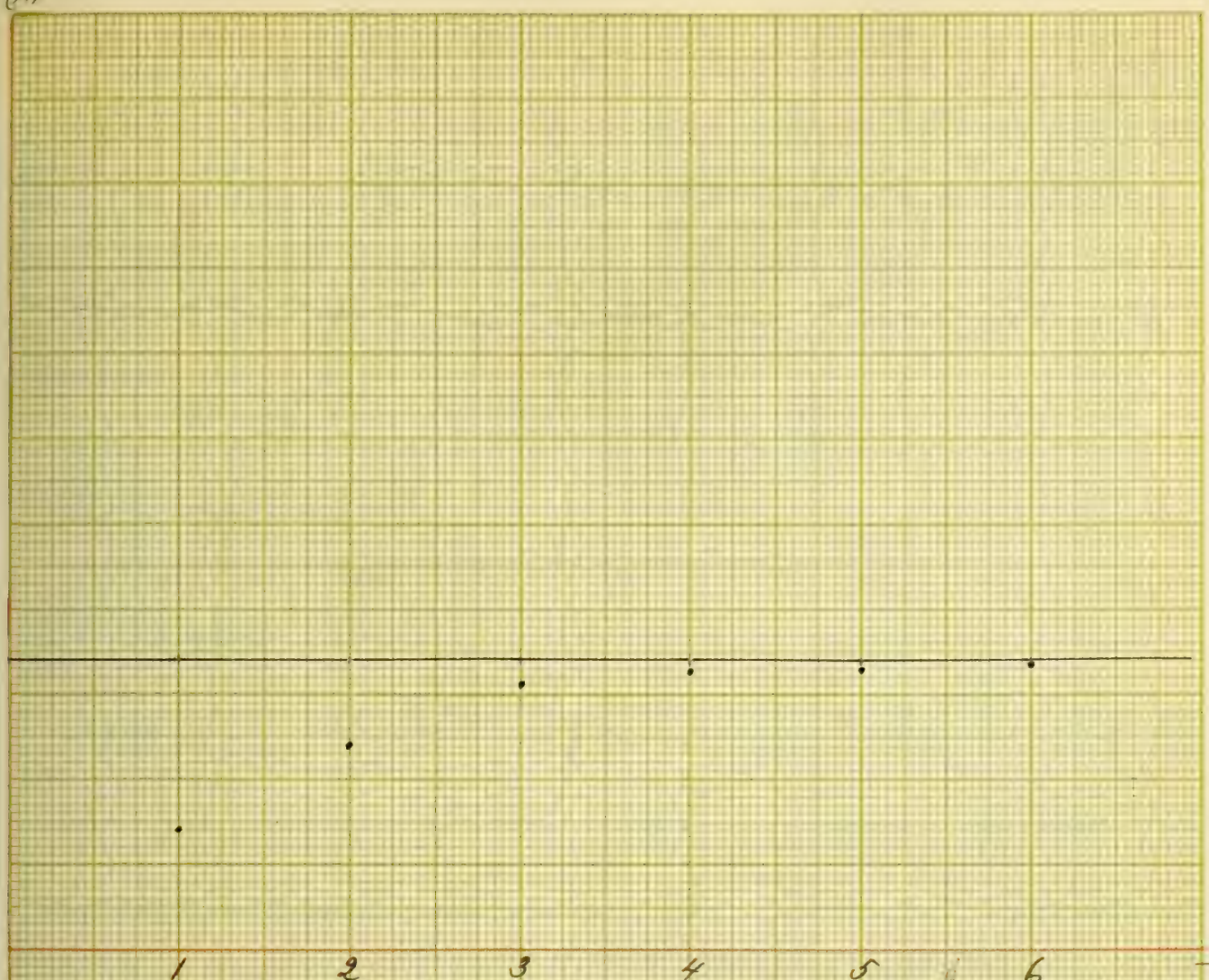
Table of values of x and the Corresponding values of $f(x)$.

x	$f(x)$
0	$(e-1) = 1.72-$
1	$(e-1)-1 = .72-$
2	$(e-1)\cdot\frac{1}{2} = 1.22-$
3	$(e-1)-\frac{1}{6} = 1.55+$
4	$(e-1)-\frac{1}{24} = 1.611+$
5	$(e-1)-\frac{1}{120} = 1.71+$
6	$(e-1)-\frac{1}{420} = 1.715+$

For all non-integral values of x

$$f(x) = (e-1) = 1.72-$$

(For plot of this function see next page)



Graph of the function

$$f(x) = \sum_{m=\infty} \frac{L}{\lfloor m} \left\{ \frac{(x - w_n)^{2/m}}{\lfloor m} \right\} \quad ; \; w_k = k.$$

Scale - 2 Squares = 1 Unit.

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Let us now take this same function again, and build up from it a function $f(x)$, discontinuous at several points at irregular intervals. Let us choose as these points now,

$$\tfrac{1}{2},\ 1,\ 2,\ \tfrac{5}{2},\ \tfrac{11}{4},\ 3,\ 4,\ \tfrac{9}{2}.$$

Moreover let

$$c_n = \frac{1}{n}.$$

We have then

$$f(x) \equiv \lim_{m=\infty} \left\{ \frac{(x-\frac{1}{2})^{2m}}{1} + \frac{(x-1)^{2m}}{2} + \frac{(x-2)^{2m}}{3} + \frac{(x-\frac{5}{2})^{2m}}{4} + \frac{(x-\frac{11}{4})^{2m}}{5} + \frac{(x-3)^{2m}}{6} + \frac{(x-4)^{2m}}{7} + \frac{(x-\frac{9}{2})^{2m}}{8} \right\}$$

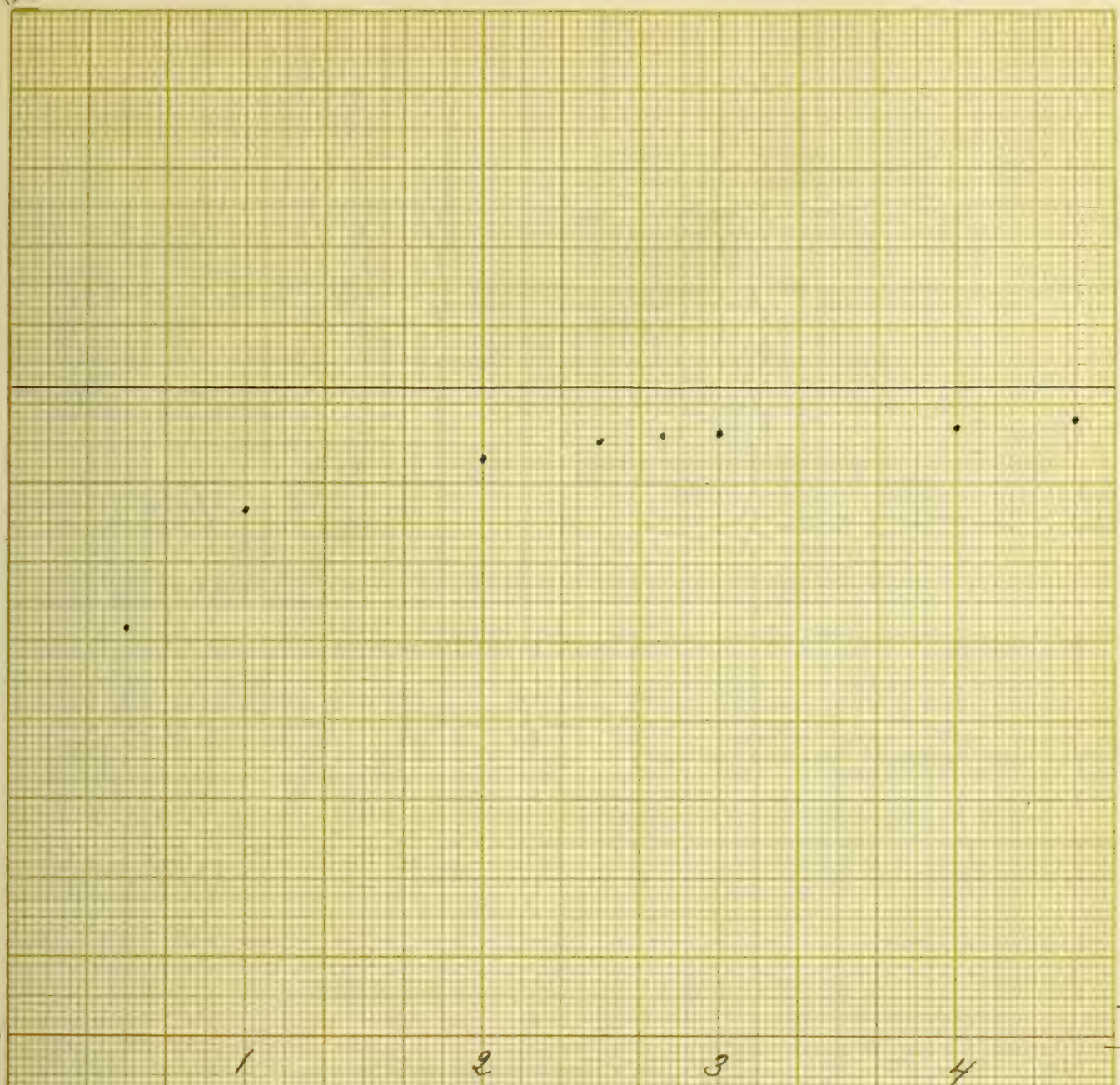
For every value of x which is not equal to one of these numbers given above we have

$$f(x) = 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8} = 2.718$$

For the other values, i.e. those belonging to the given interval we have the following values corresponding to values of x.

x	$f(x)$	x	$f(x)$
$\frac{1}{2}$	1.718	$\frac{11}{4}$	2.518
1	2.218	3	2.55
2	2.385	4	2.575
$\frac{5}{2}$	2.468	$\frac{9}{2}$	2.593

This table of values and the plot of the function given on the next page will make clear the behavior of the given function and will show the advantage of this method over that of Hankel.



Graph of the function

$$f(x) = \lim_{m=\infty} \left\{ \frac{(x-\frac{1}{2})^{2/m}}{1} + \frac{(x-1)^{2/m}}{2} + \frac{(x-2)^{2/m}}{3} + \frac{(x-\frac{5}{2})^{2/m}}{4} + \frac{(x-1\frac{1}{4})^{2/m}}{5} + \frac{(x-3)^{2/m}}{6} + \frac{(x-4)^{2/m}}{7} + \frac{(x-\frac{9}{2})^{2/m}}{8} \right\}$$

Scale - 3 Squares = 1 Unit.

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Section. IV

In this section we shall compare the results obtained by treating our function $\varphi(y)$ according to the methods of Hankel and Cantor.

If we notice carefully the nature of $f(x)$, as defined by the method of Hankel, we shall see that $f(x)$ can be made discontinuous at all integral values of x, or at all rational points. By Cantor's method, the points at which $f(x)$ is discontinuous are entirely at our disposal, for we may choose

$$a_1, a_2, a_3 \cdots a_n \cdots$$

as any Menge whatever, provided that it is abzählbar.

As to the kind of discontinuity, in certain points, by Hankel's method, the discontinuity is not of the same kind as that of $\varphi(y)$ for $y = 0$. e.g. if $\varphi(y)$ has a discontinuity of the second kind upon either side of the point $y = 0$, then $f(x)$ has such

a discontinuity upon both sides of the point $x = \gamma$, whenever γ is an odd number. By the method of Cantor, however, the discontinuity of $f(x)$ in the points

$$w_1, w_2, w_3 \cdots w_n \cdots$$

is of the same kind as that of $\varphi(y)$ for $y = 0$, except that if w_k is negative, the sides will be reversed; i.e. $f(x)$ in the point w_k will have the same characteristics on the left that $\varphi(y)$ has on the right and vice versa.

The amount of the discontinuity in the case of Hankel's method was dependent entirely upon the value of the term at the point since it depended upon when that term in the series became zero first. By the method of Cantor, since we may arrange our Merge

$$w'_1, w'_2, w'_3 \cdots w'_n \cdots$$

as we choose, we are enabled to make the discontinuity in any point as large or as small as we choose, except that the number of points in which the discontinuity is greater than some

fixed, finite number cannot be infinite. This last restriction, however, applies to Hankel's method also, so that in either case the number of terms in which the growth of the function is greater than e^{σ} is always finite. Furthermore, neither method will give us a function which is discontinuous at every point.

Section V

9

In this section we shall discuss, by a method somewhat more simple than the one previously used, two kinds of functions. These two kinds of functions are defined as follows;

(a) Those functions which are finite and determined between the limits -1 and $+1$, the end points included, are discontinuous at the point $y=0$, and are <u>not</u> equal to a constant for $y \neq 0$, y being the variable.

e.g. $\dfrac{1}{1+e^{1/y}}$

(b) Those functions which are finite and determined between the limits given above, but are discontinuous at the point $y=0$, and <u>have</u> a constant value for $y \neq 0$.

e.g. $\lim_{t=0} \left(\dfrac{y^2}{y^2+t^2}\right)$

From the functions under (a) we shall build up new functions which are discontinuous at a finite number of points. From those under (b) we shall obtain functions which are

discontinuous at (1) every point, and (2) every point but one.

(a). We now define our $\varphi(y)$ as a function of y which is finite and determined between the limits $+1$ and -1, the end points included, but is discontinuous at the point $y=0$, and is not equal to a constant for $y \neq 0$.

We now define $f(x)$ as follows,

$$f(x) \equiv \varphi(\sin^m \pi x)$$

where m is at our disposal, except that it must not become greater than some fixed finite number q, and when once fixed remains constant. Since by hypothesis $\varphi(y)$ is continuous at every point except $y=0$ and is discontinuous at $y=0$, $f(x)$ will be discontinuous when $\sin^m \pi x = 0$, i.e. when $\sin^m x = k$, an integer. Within any given interval there can only be a finite number of values of x which will satisfy this relation, depending upon the value of m, unless we let m approach infinity. But this is contrary to our definition of m. Moreover if we should let m approach infinity our function $f(x)$ becomes indeterminate

in the irrational points for there we should have

$$f(x) = \varphi(\sin \infty)$$

where

$$\sin \infty \neq 0$$

Hence the value of $f(x)$ would oscillate between the maximum and minimum values of $\varphi(y)$ between -1 and 0, and 0 and $+1$.

Therefore we have only a finite number of rational points for which

$$f(m) = k_q$$

and $f(x)$ is discontinuous only at a finite number of rational points.

As an illustration of this method of condensation let us take the function cited on page 59. We have then,

$$\varphi(y) = \frac{1}{1+e^{\frac{1}{y}}}.$$

Moreover we must define $\varphi(y)$ in the point zero, so we now define

$$\frac{1}{1+e^{\frac{1}{y}}} = 0$$

for $y = 0$.

It may be easily seen that $\varphi(y)$

is continuous at all points $\neq 0$ in such $y \neq 0$. But at the point $y = 0$ we have

$$\lim_{\delta=0} \left(\frac{1}{1 + e^{(0+\delta)}} \right) = \frac{1}{1+\infty} = 0$$

and

$$\lim_{\delta=0} \left(\frac{1}{1 + e^{(0-\delta)}} \right) = \frac{1}{1+\frac{1}{\infty}} = 1.$$

i.e. $\varphi(y)$ is continuous to the right of the point $y = 0$ but discontinuous to the left of that point.

Let us now form $f(x)$. We have then

$$f(x) \equiv \frac{1}{1 + e^{(2\sin m\pi x)}}$$

where m is some fixed, finite number. Now let $m = 2$. Then

$$f(x) \equiv \frac{1}{1 + e^{2\sin 2\pi x}}$$

For this value of m

$$\text{Lt } x = K$$

becomes

$$2x = K$$

which holds for

$$x = \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3 \text{ ----------}$$

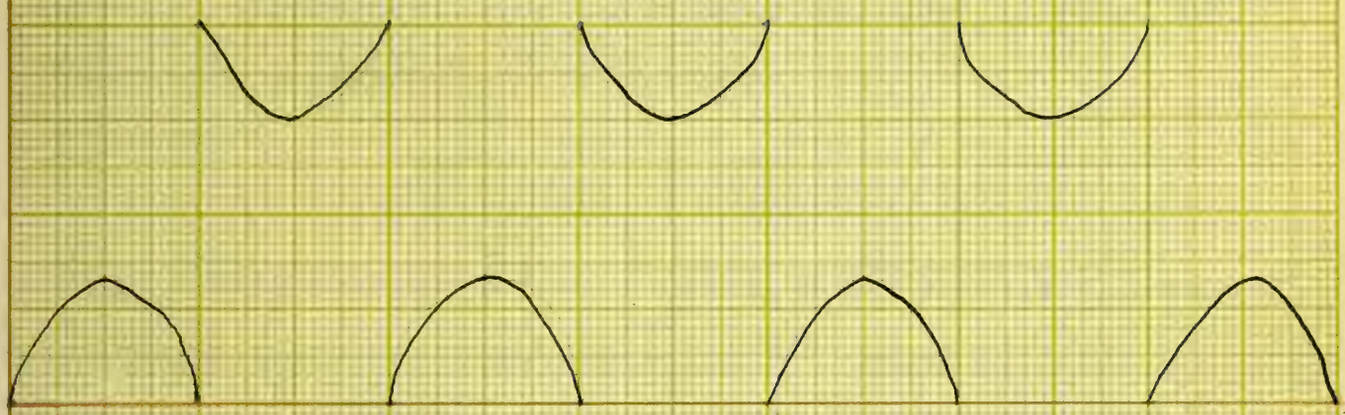
i.e. $f(x)$ will be discontinuous at points at a distance $\frac{1}{2}$ apart, and there can be only a finite number of these points within any given interval.

A table of values is given below and a part of a portion of the curve will be plotted upon the next page.

Table of Values of x and the Corresponding Values of f(x).

X.	f(x).
0	0
1/4	1/3
1/2 − δ	0
1/2 + δ	1
3/4	3/4
1 − δ	1
1 + δ	0
1 1/4	1/3
3/2 − δ	0
3/2 + δ	1

P(x)



Sketch
of the function

$$f(x) \equiv \frac{1}{1+e^{\sin 2\pi x}}$$

Scale - 4 Squares = 1 Unit.

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(b) In this part of the discussion we define $\varphi(y)$ as a function of y which is equal to zero for $y = 0$ and to some constant A for $y \neq 0$.

(1) We wish now to build up from this function $\varphi(y)$ a new function $f(x)$ which shall be discontinuous at every point. In order to do this we define $f(x)$ as follows;

$$f(x) \equiv \lim_{n=\infty} \varphi(\sin^{2n} \pi x)$$

Since $\sin^{2n} \pi x \neq 0$ for all irrational values of x (See page 9) and $\varphi(y) = A$ for $y \neq 0$ we have

$$f(x) = A$$

for all irrational values of x.

For the rational values i.e. where $x = p/q$, we know that

$$\lim_{n=\infty} (\sin^{2n} \pi x) = 0$$

Then for these points, since $\varphi(y) = 0$ for $y = 0$,

$$f(x) = 0$$

Therefore, since $f(x) = A$ for all irrational, and $f(x) = 0$ for all rational points it must be discontinuous at every point.

Example.

Since

$$\varphi(y) = \lim_{t=0}\left(\frac{y^2}{y^2+t^2}\right)$$

We have then $\varphi(y)=0$ for $y=0$, and $\varphi(y)=1$ for $y\neq 0$. Making the substitution

$$y = \lim_{m=\infty} \sin m\pi x$$

and forming $f(x)$ we have

$$f(x) = \lim_{n=\infty}\left\{\lim_{t=0}\left(\frac{\sin^2 m\pi x}{\sin^2 m\pi x + t^2}\right)\right\}$$

Since

$$\lim_{n=\infty}(\sin m\pi x) \neq 0$$

for all irrational values of x, we have

$$f(x) \equiv 1$$

for all irrational values of x. For the rational values we have

$$\lim_{n=\infty}(\sin m\pi \tfrac{p}{q}) = 0$$

and hence

$$f(x) = 0$$

for all rational values of x.

Therefore, since

$$f(x) = 1$$

for irrational values of x, and

$$f(x) = 0$$

for rational values, $f(x)$ is totally discontinuous, i.e. discontinuous at every point. (For plot see page 70).

(2) In this part of our discussion we wish to build up from our given function $\varphi(y)$, defined under (6), a new function $f(x)$ which shall be discontinuous at every point but one.

For this purpose we now define $f(x)$ as follows;

$$f(x) = \lim_{n=\infty} x\, \varphi(\sin n! \pi x).$$

Following out the same line of reasoning as in (1) we find that

$$f(x) = Ax$$

for all irrational points and

$$f(x) = 0$$

for all rational points. Then, since $f(x)$ approaches zero as x approaches zero in both of these sets of points, it is continuous at the point $x=0$, tho discontinuous at every other point. i. e. $f(x)$ is discontinuous at every point but one.

Example.

Let us take the same function of y as used in the first part of this discussion, namely

$$\varphi(y) \equiv \lim_{t=0} \left(\frac{y^2}{y^2+t^2}\right)$$

Then $\varphi(y) = 0$ for $y=0$, and $\varphi(y) = 1$ for $y \neq 0$,

Making the substitution

$$y = \sin^{2m} \pi x$$

and passing to $f(x)$ we have

$$f(x) \equiv \lim_{n=\infty} \left\{ x \left(\lim_{t=0} \frac{\sin^{2m} \pi x}{\sin^{2m} \pi x + t^2} \right) \right\}$$

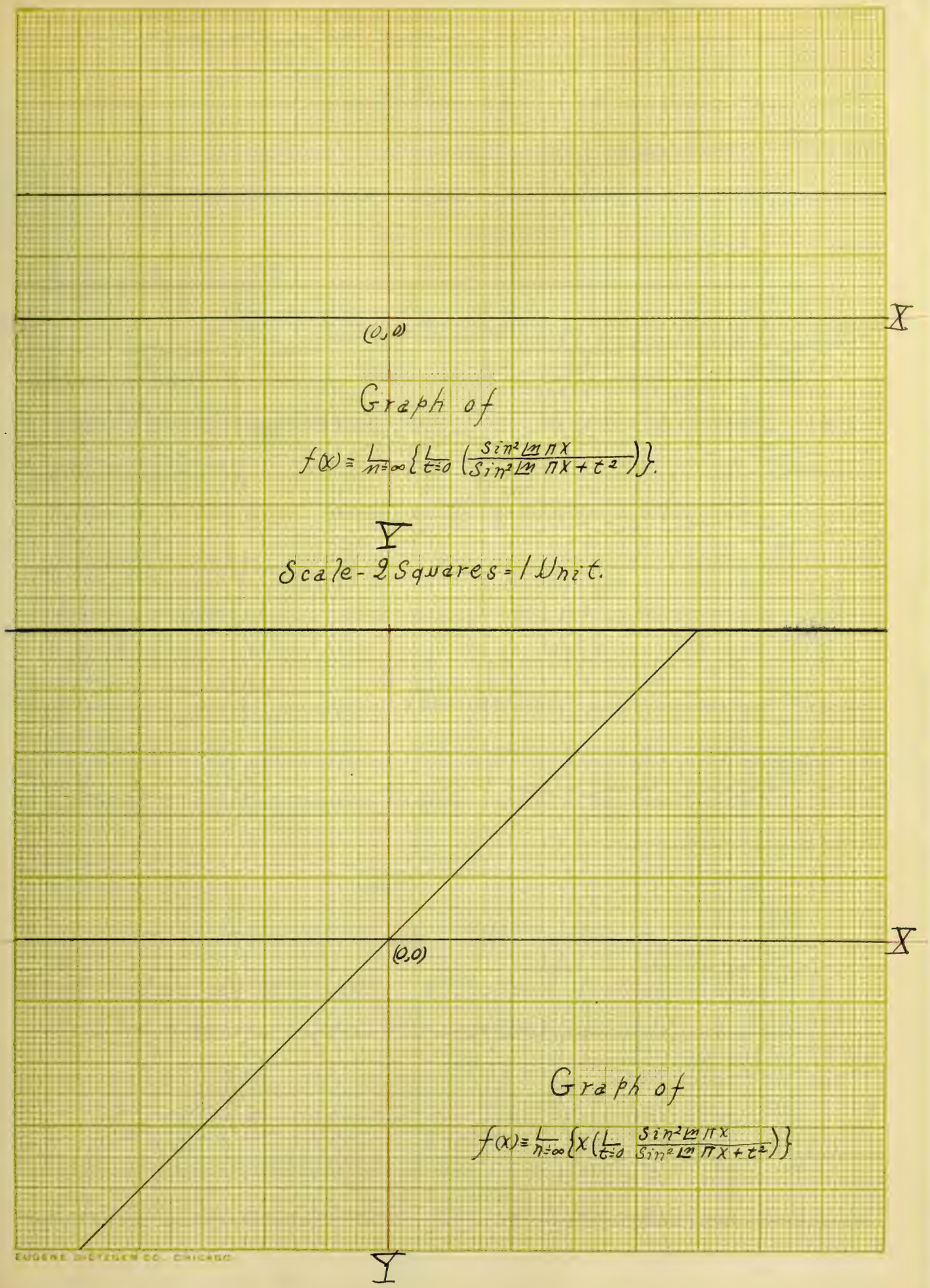
For the irrational values of x, by the same reasoning as that used on page , we have

$$f(x) = x$$

For the rational values we have as before

$$f(x) = 0$$

It may be easily seen from this and from the plot on the next page that $f(x)$ is discontinuous at every point except $x = 0$, and is continuous at that point.



Graph of

$$f(x) \equiv \lim_{n=\infty} \left\{ \lim_{t=0} \left(\frac{\sin^2 \underline{m} \pi x}{\sin^2 \underline{m} \pi x + t^2} \right) \right\}.$$

Scale - 2 Squares = 1 Unit.

Graph of

$$f(x) \equiv \lim_{n=\infty} \left\{ x \left(\lim_{t=0} \frac{\sin^2 \underline{m} \pi x}{\sin^2 \underline{m} \pi x + t^2} \right) \right\}$$

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Chapter II.

Points at Which a Function Has No Derivative.

Section VI

In this section we shall treat our function $\varphi(y)$ according to the method of **section I**. Two kinds of functions will be discussed; (a) those which have everywhere a fixed and finite derivative except at the point $y=0$, and (b), those which have in certain points, as y approaches indefinitely near to zero, a derivative which is fixed, but may be larger than any given finite value g.

(a)

Let us take $\varphi(y)$ now as a function of y, which, for all values of y between -1 and $+1$, the end points included, is finite and continuous and which, at every point within this interval, at most $y=0$ excepted, has a fixed and finite derivative, which

is less than some finite number β, and also let $\varphi(y) = 0$ for $y = 0$.

Then the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi x)}{(\lfloor n)^2}$$

will be a finite and continuous function for all values of x. This follows from the fact that $\sin \lfloor n \pi x$ can never exceed the limits -1 to $+1$, and $\varphi(y)$ being finite and continuous within this interval, the maximum values of the terms defining $f(x)$ form a series in which the terms are less than the corresponding terms of the series

$$g' \sum_{n=1}^{\infty} \left(\frac{1}{\lfloor n}\right)^2$$

where g' is the maximum value of $\varphi(y)$ in the given interval. $\sum_{n=1}^{\infty} \left(\frac{1}{\lfloor n}\right)^2$ is a known convergent series. Therefore according to the theorem given on page 7

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi x)}{(\lfloor n)^2}$$

is a continuous function.

Moreover, for irrational x, the series formed from the sum of the derivatives of the terms of the series given above, namely,

$$\Pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{(\lfloor n)^2} \cos \lfloor n \pi x .$$

will also be a uniformly convergent series, since none of these terms are greater than g, and

$$g \sum_{n=1}^{\infty} \frac{1}{(\lfloor n)^2}$$

is a convergent series. Therefore, for irrational x the derived function of $f(x)$ is equal to

$$f'(x) \equiv \sum_{n=1}^{\infty} \Pi \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

since

$$\Pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

is a uniformly convergent series for every x for which we have defined $f(x)$, i.e. for every irrational point. (See

If $\varphi(y)$ has a fixed derivative in the point $y = 0$ also, then

$$f'(x) \equiv \Pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

will be the derivative of $f(x)$ at every point since our series

$$\Pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

will then be uniformly convergent for all values of x in any finite interval.

Now let us take up the case when $\varphi(y)$ has no fixed derivative at the point $y = 0$: i.e.

$$\frac{\varphi(0\pm\delta)-\varphi(0)}{\delta} = \frac{\varphi(0\pm\delta)}{\delta} = \frac{\varphi(\pm\delta)}{\delta}$$

has no fixed limit, but can never exceed some definite, finite value; i.e. it oscillates between finite limits.

We have already seen that for irrational values of x, the series of the derivatives of the individual terms, viz:-

$$\pi \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

is uniformly convergent and is therefore equal to the derived function of $f(x)$. Hence $f(x)$ has in all irrational points, a fixed and finite derivative, namely,

$$f'(x) = \sum_{n=1}^{\infty} \frac{\varphi'(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x$$

Now let us examine this series $f'(x)$ in the rational points.

$$\lim_{\delta=0} \left\{ \frac{f(\frac{h}{k}\pm\delta) - f(\frac{h}{k})}{\delta} \right\} = \pi \sum_{n=1}^{n=\alpha-1} \frac{\varphi'(\sin \lfloor n \, h/k \, \pi}{\lfloor n} \cos \lfloor n \, \pi \, h/k$$

$$+ \sum_{n=\alpha}^{\infty} \lim_{\delta=0} \frac{\varphi(\pm \sin \lfloor n \pi \delta)}{(\lfloor n)^2 \delta}$$

(See pages 11-12)

In this case, as in the one referred to, the sign of $\sin \lfloor n \pi \delta$ depends upon the sign of δ and upon whether n is even or odd. We know that the same rea-

soning applies to the first summation on the right of the sign of equality, as to the case of the irrational number since $\sin \lfloor m \pi \frac{h}{q} \neq 0$ in each term. This first summation, then, represents the derivative of $\sum_{m=1}^{\alpha-1} \frac{\varphi(\sin \lfloor m \pi \frac{h}{q})}{(\lfloor m \rfloor)^2}$

It is sufficient then for us to examine the second summation, namely;

$$\sum_{m=\alpha}^{\infty} \lfloor \frac{\varphi(\sin \lfloor m \pi \delta)}{(\lfloor m \rfloor)^2 \delta}$$

We have under this three cases similar to those discussed on pages 12-13 under discontinuity, namely;

(a)

$$\lfloor_{\delta=0} \left\{ \frac{f(\frac{h}{q}+\delta) - f(\frac{h}{q})}{\delta} \right\} = \Pi \sum_{m=1}^{\alpha-1} \frac{\varphi'(\sin \lfloor m \pi \frac{h}{q})}{\lfloor m} \cos \lfloor m \pi \frac{h}{q} + \sum_{m=\alpha}^{\infty} \lfloor_{\delta=0} \frac{\varphi(\sin \lfloor m \pi \frac{h}{q})}{(\lfloor m \rfloor)^2 \delta} \quad \because \lfloor \alpha = 2k'q$$

$$\lfloor_{\delta=0} \left\{ \frac{f(\frac{h}{q}-\delta) - f(\frac{h}{q})}{\delta} \right\} = \Pi \sum_{m=1}^{\alpha-1} \frac{\varphi'(\sin \lfloor m \pi \frac{h}{q})}{\lfloor m} \cos \lfloor m \pi \frac{h}{q} + \sum_{m=\alpha}^{\infty} \lfloor_{\delta=0} \frac{\varphi(-\sin \lfloor m \pi \delta)}{(\lfloor m \rfloor)^2 \delta} \quad \because \lfloor \alpha = 2k'q$$

(b).

$$\lfloor_{\delta=0} \left\{ \frac{f(\frac{h}{q}+\delta) - f(\frac{h}{q})}{\delta} \right\} = \Pi \sum_{m=1}^{\alpha-1} \frac{\varphi'(\sin \lfloor m \pi \frac{h}{q})}{(\lfloor m \rfloor)^2} \cos \lfloor m \pi \frac{h}{q} + \frac{1}{(\lfloor \alpha \rfloor)^2} \lfloor_{\delta=0} \frac{\varphi(-\sin \lfloor m \pi \delta)}{\delta} + \sum_{m=\alpha+1}^{\infty} \lfloor_{\delta=0} \frac{\varphi(\sin \lfloor m \pi \delta)}{(\lfloor m \rfloor)^2 \delta} \quad \because \lfloor \alpha = (2k+1) q/p$$

$$\lim_{\delta=0}\left\{\frac{f(\tfrac{h}{q}-\delta)-f(\tfrac{h}{q})}{\delta}\right\}=\Pi\sum_{m=1}^{\alpha-1}\frac{\varphi'(\sin m\pi\, h/q)}{m}\cos m\pi\, h/q$$
$$+\frac{1}{(\alpha)^2}\lim_{\delta=0}\frac{\varphi(\sin\alpha\pi\delta)}{\delta}$$
$$+\sum_{m=\alpha+1}^{\infty}\lim_{\delta=0}\frac{\varphi(-\sin m\pi\delta)}{(m)^2\delta}\qquad \alpha=(2k'+1)q/h$$

(c).

$$\lim_{\delta=0}\left\{\frac{f(\tfrac{h}{q}+\delta)-f(\tfrac{h}{q})}{\delta}\right\}=\Pi\sum_{m=1}^{\alpha-1}\frac{\varphi'(\sin m\pi\, h/q)}{(m)}\cos m\pi\, h/q$$
$$+\left\{\frac{1}{(\alpha)^2}+\frac{1}{(\alpha+1)^2}\right\}\lim_{\delta=0}\frac{\varphi(-\sin\alpha\pi\delta)}{\delta}$$
$$+\sum_{m=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(\sin m\pi\delta)}{(m)^2\delta}\qquad \alpha=(2k'+1)q/h$$

$$\lim_{\delta=0}\left\{\frac{f(\tfrac{h}{q}-\delta)-f(\tfrac{h}{q})}{\delta}\right\}=\Pi\sum_{m=1}^{\alpha-1}\frac{\varphi'(\sin m\pi\, h/q)}{m}\cos m\pi\, h/q$$
$$+\left\{\frac{1}{(\alpha)^2}+\frac{1}{(\alpha+1)^2}\right\}\lim_{\delta=0}\frac{\varphi(-\sin\alpha\pi\delta)}{\delta}$$
$$+\sum_{m=\alpha+2}^{\infty}\lim_{\delta=0}\frac{\varphi(-\sin m\pi\delta)}{(m)^2\delta}\qquad \alpha=(2k'+1)q/h$$

Bearing in mind the fact that $\lim_{\delta=0}\sin m\pi\delta=0$, we may see from these equations that the following statements are true;

In case (a), namely when $\alpha=2k'q/h$ (See page 10), the derivative of $f(x)$ in the point $x=h/q$, will exist or will not exist according as the derivative of $\varphi(y)$ exists or does not exist at the point $y=0$, and will correspond in all particulars except that, if it does exist, it will be

multiplied by a constant factor.

In cases (b) and (c) we have $\lim_{\delta=0} \frac{\varphi(\sin \lfloor m \pi \delta)}{\delta}$ and $\lim_{\delta=0} \frac{\varphi(-\sin \lfloor m \pi \delta)}{\delta}$ entering into the equation both for a negative and for a positive increment. Hence if $\lim_{\delta=0} \frac{\varphi(\delta)}{\delta}$ or $\lim_{\delta=0} \frac{\varphi(-\delta)}{\delta}$ does not exist, or if either vary between finite limits, then the derivative of $f(x)$ in the point $x = \frac{h}{g}$ will behave in the same manner for both a positive and a negative increment. This can be seen more plainly if we write our expression for the derivative of $f(x)$ in the point $x = \frac{h}{g}$ in the following form

$$\lim_{\delta=0}\left\{\frac{f(\frac{h}{g}\pm\delta)-f(\frac{h}{g})}{\delta}\right\} = \pi \sum_{m=1}^{m=\alpha-1} \frac{\varphi'(\sin \lfloor m \pi \delta)}{\lfloor m} \cos \lfloor m \pi \frac{h}{g}$$

$$+ A\left\{\lim_{\delta=0} \frac{\varphi(\mp\delta)}{\delta}\right\} + B\left\{\lim_{\delta=0} \frac{\varphi(\pm\delta)}{\delta}\right\}$$

Since the A and the B are constants representing the constants in our equations (b) and (c) and also the first summation on the right of the sign of equality has a fixed value, if either $\frac{\varphi(-\delta)}{\delta}$ or $\frac{\varphi(+\delta)}{\delta}$ have no definite limit the whole expression taken together can have no limit and our statement made above is proved.

Observe, in all three cases (a), (b) and (c), if $\frac{\varphi(s)}{s}$ and $\frac{\varphi(s)}{s}$ both have limits but these limits are different, then $f(x)$ will show the same characteristics for positive and negative increments at the point $x = \eta$, i.e. its derivative will exist both to the right and to the left of the point but the limits of these two will not be equal.

We may now sum up our results as follows;

If $\varphi(y)$ is a function of y, which, for all values of y between -1 and $+1$, the end points included, is finite and determined and continuous, and for every point of this interval, except at the point $y = 0$, has a fixed and finite derivative, then

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \rfloor \pi x)}{(\lfloor n \rfloor)^2}$$

will have the following characteristics, namely;

In all irrational points $f(x)$ will have a fixed and definite derivative. In all rational points for which k is even (see page 21) the derivative

at $f(x)$ will be of the same nature as that of $\varphi(y)$ in the point $y=0$. In the rational points for which x is odd, if the derivative of $\varphi(y)$ has a limit upon both sides of the point $y=0$, but these limits are different, $f(x)$ will have the same characteristics in regard to its derivative, but if the derivative of $\varphi(y)$ oscillates upon either side of the point $y=0$, then the derivative of $f(x)$ will oscillate upon both sides of the points in question.

Example.

Given;

$$\varphi(y) \equiv y \sin 1/y. \qquad \text{(See page 83)}$$

This is a function of y which is continuous between the limits

$$-1 \geqq y \geqq +1$$

and, examining its derivative we have

$$\frac{\varphi(y \pm \delta) - \varphi(y)}{\delta} \equiv \frac{(y \pm \delta)(\sin \frac{1}{y \pm \delta}) - y \sin \frac{1}{y}}{\delta}$$

which is fixed for $y \neq 0$. When $y = 0$ we have

$$\frac{\varphi(0 \pm \delta) - \varphi(0)}{\delta} = \frac{(0 \pm \delta) \sin \frac{1}{0 \pm \delta} - 0 \cdot \sin \frac{1}{0}}{\delta}$$

cf.

$$\frac{\varphi(t\delta)}{\delta} = t\delta \sin\frac{1}{t\delta} - \delta \sin\frac{1}{\delta}.$$

We know that $\sin\frac{1}{\delta}$ oscillates between the limits -1 and $+1$, as δ approaches zero, hence the derivative of $\varphi(t)$ for $t=0$, oscillates between the limits -1 and $+1$, for $\delta = 0$.

Forming $f(x)$ we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\sin \lfloor n \pi x}{(\lfloor n)^2} \sin\frac{1}{\sin \lfloor n \pi x}$$

Since $\varphi(t)$ has a fixed, finite derivative for $t \neq 0$, and $\sin \lfloor n \pi x \neq 0$ for all irrational values of x, $f(x)$ must have a fixed and finite derivative for all irrational values of x. Moreover this derivative will be

$$f'(x) \equiv \pi \sum_{n=1}^{\infty} \frac{\cos \lfloor n \pi x \sin\frac{1}{\sin \lfloor n \pi x} + \frac{1}{\sin \lfloor n \pi x} \cos\frac{1}{\sin \lfloor n \pi x}}{\lfloor n}$$

For the rational numbers we have

$$\lim_{\delta = 0} \left\{ \frac{f(\frac{\alpha}{\beta} \pm \delta) - f(\frac{\alpha}{\beta})}{\delta} \right\}$$

$$= \pi \sum_{n=1}^{\alpha-1} \frac{\cos \lfloor n \pi \frac{\alpha}{\beta} \sin\frac{1}{\sin \lfloor n \pi \frac{\alpha}{\beta}} + \frac{1}{\sin \lfloor n \pi \frac{\alpha}{\beta}} \cos\frac{1}{\sin \lfloor n \pi \frac{\alpha}{\beta}}}{\lfloor n}$$

$$+ \lim_{\delta = 0} \sum_{n=\alpha}^{\infty} \frac{\sin \lfloor n \pi (\frac{\alpha}{\beta} \pm \delta) \sin\frac{1}{\sin \lfloor n \pi (\frac{\alpha}{\beta} \pm \delta)} - \sin \lfloor n \pi \frac{\alpha}{\beta} \sin\frac{1}{\sin \lfloor n \pi \frac{\alpha}{\beta}}}{(\lfloor n)^2 \delta}$$

$$= g + \lim_{\delta=0} \sum_{n=\alpha}^{\infty}{}' \frac{\sin \lfloor n \rfloor \pi (k/g + \delta) \cdot \sin \frac{1}{\sin \lfloor n \rfloor \pi (k/g + \delta)}}{(\lfloor n \rfloor)^2 \delta}$$

where g represents the value of the sum of the derivatives of all the terms in which

$$p n \neq k g$$

and therefore

$$\sin \lfloor n \rfloor \pi \, k/g \neq 0,$$

and the last term in the second summation drops out since

$$\sin \lfloor n \rfloor \pi \, k/g = 0$$

in each of those terms. Our expression may also be written in the form

$$\lim_{\delta=0} \left\{ \frac{\pm (k/g \pm \delta) - f(k/g)}{\delta} \right\} = g + \lim_{\delta=0} \sum_{n=\alpha}^{\infty} \frac{\pm \sin \lfloor n \rfloor \pi \delta}{(\lfloor n \rfloor)^2 \delta} \sin \frac{1}{\pm \sin \lfloor n \rfloor \pi \delta}$$

$$= g + \sum_{n=\alpha}^{\infty} \pm \frac{\pi}{\lfloor n \rfloor} \sin \frac{1}{0}$$

$$= g + \pi \sum_{n=\alpha}^{\infty} \frac{1}{n} \cdot \sin \frac{1}{0}.$$

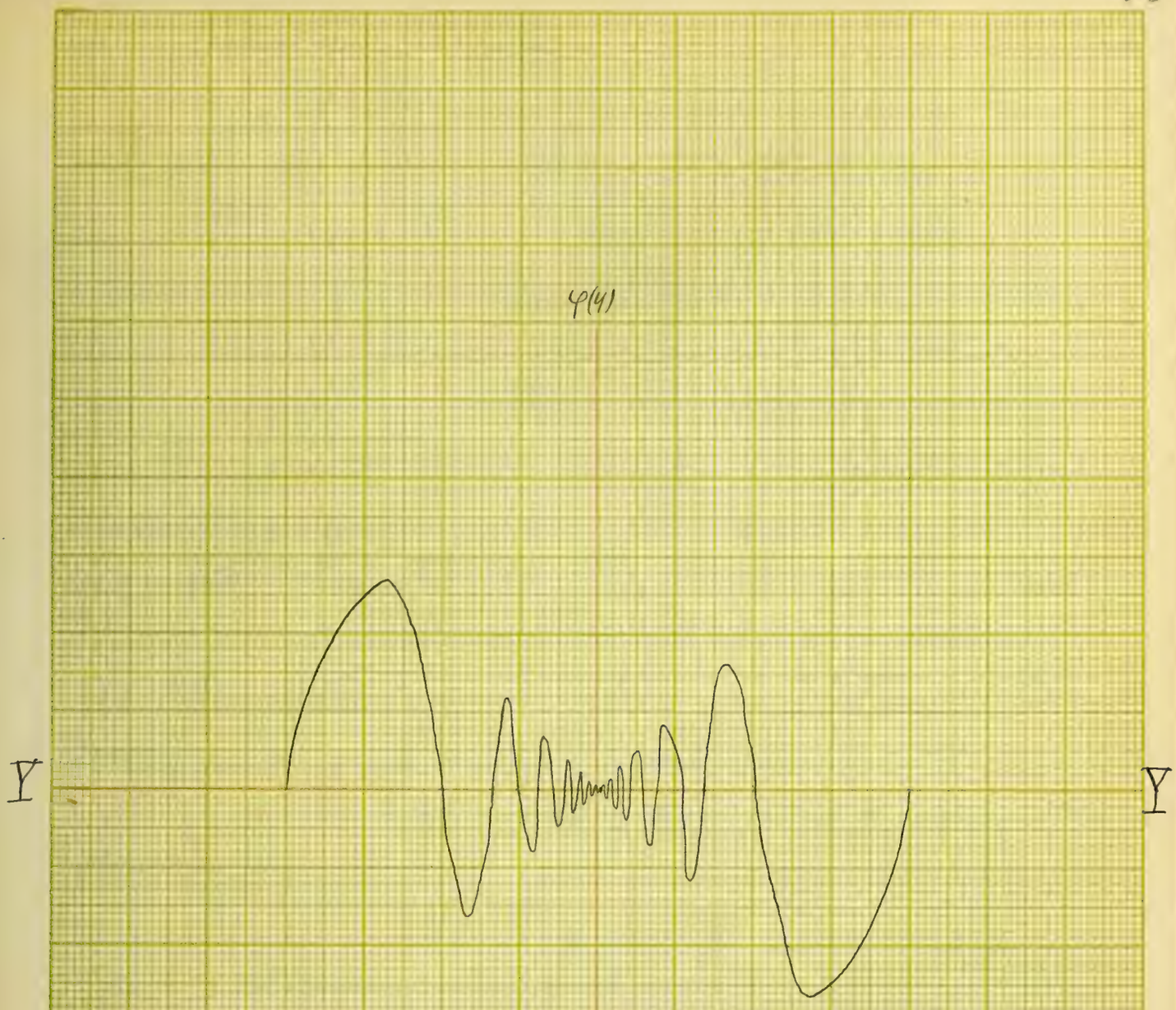
Since $\sin \frac{1}{0}$ oscillates between -1 and $+1$, the derivative of $f(x)$ in the point k/g between the limits

$$g + \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n \rfloor}$$

and

$$g - \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n \rfloor}$$

Thus we see that just as $\varphi(y)$ oscillates in the neighborhood of the point $y=a$, $f(x)$ oscillates in the neighborhood of every rational point, but is fixed for all irrational points.



Rough Sketch
showing the behavior of
the function

$\varphi(y) \equiv y \sin 1/y$
in the neighborhood
of the point $y = 0$.

Scale - 4 Squares = 1 Unit.

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(6).

Let $\varphi(y)$ now be a function of y, which, for all values of y between the limits -1 and $+1$, the end points included, is finite, determined, continuous, and for every point of the interval except $y=0$ has a fixed and definite derivative, but as y approaches indefinitely near to zero, in certain points this derivative becomes greater than any fixed finite number g. At the point $y=0$, $\varphi'(y)$ may have the following characteristics,

(1) $\varphi'(y)$ may be finite and fixed for $y=0$. e.g.

$$\varphi(y) = Ay^2 \sin 1/y$$

Then $\varphi'(0) = 0$ and $\varphi'(y) = A \cos 1/y$.

(2). $\varphi'(y)$ may be infinite and of determined sign e.g.

$$\varphi(y) = \log \sin |y|$$

Then $\varphi'(y) = \cot |y|$ and $\varphi'(0) = +\infty$

(3) $\varphi'(y)$ may be infinite and of undetermined sign for $y=0$.

e.g. $\varphi(y) = \log \sin y$

Then $\varphi'(y) = \cot y$ $\varphi'(0) = \pm\infty$.

(4) $\varphi'(y)$ may be wholly indeter-

oscillate in the point $y = 0$.

e.g. $\varphi(y) = \cos 1/y$

Then

$$\varphi'(y) = \frac{\sin 1/y}{y^2}$$

which is wholly indeterminate in the point $y = 0$, ranging from $-\infty$ to $+\infty$.

If, now, we define $f(x)$ as before, i.e.

$$f(x) \equiv \sum_{n=1}^{n=\infty} \frac{\varphi(\sin n! \pi x)}{(n!)^2}$$

we can no longer say any thing about the behavior of the derivative of $f(x)$ in the irrational points. This follows from the fact that for irrational values of x, $n!x$ approaches more nearly an integeral value as n grows larger, and hence the derivatives of the individual terms no longer remain less than some finite number g for the higher terms in the series and the process of taking the derivatives, term by term is no longer applicable. It therefore remains for us to examine the behavior of the derivative of $f(x)$ in the rational points.

Let us take first a function $\varphi(y)$ which has a fixed and finite

derivative in the point $y = \delta$. Since this is true every term of the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi \, x/y \rfloor)}{(\lfloor n \rfloor)^2}$$

will have a fixed derivative which will be a term of a new series

$$\pi \sum_{n=1}^{\infty} \frac{\varphi(\sin \lfloor n \pi \, x/y \rfloor)}{\lfloor n \rfloor} \cos \lfloor n \pi \, x/y \rfloor$$

This new series will be equal to the derivative of $f(x)$, provided we can show that

$$\sum_{n=1}^{m} \left\{ \frac{\varphi(\sin \lfloor n \pi (x/y + \delta) \rfloor) - \varphi(\sin \lfloor n \pi \, x/y \rfloor)}{(\lfloor n \rfloor)^2 \delta} - \pi \frac{\varphi'(\sin \lfloor n \pi \, x/y \rfloor)}{\lfloor n \rfloor} \cos \lfloor n \pi \, x/y \rfloor \right\}$$

$$\frac{R_m(x/y)}{\delta} = \sum_{n=m+1}^{\infty} \frac{\varphi(\sin \lfloor n \pi \, x/y \rfloor)}{(\lfloor n \rfloor)^2 \delta}$$

and

$$\frac{R_m(x/y + \delta)}{\delta} = \sum_{n=m+1}^{\infty} \frac{\varphi(\sin \lfloor n \pi (x/y + \delta) \rfloor)}{(\lfloor n \rfloor)^2 \delta}$$

may each be made less than some arbitrarily small positive number σ, by making m sufficiently large and δ sufficiently small. *

Let m now be the first number for which

$$\lfloor m \rfloor > \frac{1}{|\delta| \delta}$$

Then

$$(\lfloor m \rfloor)^2 > \frac{1}{|\delta|^2 \delta}$$

and

$$\frac{1}{(\lfloor m \rfloor)^2} < |\delta|^2 \delta$$

* See Dini - § 103. page 152.

Furthermore, let g be the maximum value of $\varphi(u)$ between the limits -1 and $+1$, the end points included. Then, we can certainly write

$$R_m\left(\frac{w}{g}\right) < g \sum_{n=m+1}^{\infty} \frac{1}{(n)^2}$$

or

$$R_m\left(\frac{w}{g}\right) < g \frac{1}{(m)^2}$$

for it can be easily proven that

$$\frac{1}{(m)^2} > \sum_{n=m+1}^{\infty} \frac{1}{(n)^2}.$$

Substituting now our value of $\frac{1}{(m)^2}$ in terms of δ, we have

$$R_m\left(\frac{w}{g}\right) < g|\delta|^{2\theta}$$

or

$$\frac{R_m\left(\frac{w}{g}\right)}{\delta} < g|\delta|^{2\theta-1}$$

In like manner we may also obtain

$$\frac{R_m\left(\frac{w}{g}+\delta\right)}{\delta} < g|\delta|^{2\theta-1}$$

Since g may be taken greater than $1/2$, we can always make this as small as we choose and the two conditions

$$\frac{R_m\left(\frac{w}{g}+\delta\right)}{\delta} < \sigma_1$$

and

$$\frac{R_m\left(\frac{w}{g}\right)}{\delta} < \sigma_2$$

Now, since m is finite, and $\varphi(u)$

has a derivative at every point, which is fixed even tho it becomes very large as y approaches zero, the series

$$\sum_{n=1}^{\infty}\left\{\frac{\varphi(\sin 2^n\pi(\frac{x}{y}+\delta)) - \varphi(\sin 2^n\pi\frac{x}{y})}{(2^n)^2\delta} - \pi\varphi'\left(\frac{\sin 2^n\pi\frac{x}{y}}{2^n}\right)\cos 2^n\pi\frac{x}{y}\right\}$$

remain less than τ_3. Thus all the conditions are fulfilled and we can apply our differentiation term by term to the series

$$f(x) \equiv \sum_{n=1}^{\infty}\frac{\varphi(\sin 2^n\pi\frac{x}{y})}{(2^n)^2}$$

We can, therefore, write our derivative thus:

$$\lim_{\delta=0}\left\{\frac{f(\frac{x}{y}+\delta) - f(\frac{x}{y})}{\delta}\right\} = \pi\sum_{n=1}^{\alpha}\varphi'\left(\frac{\sin 2^n\pi\frac{x}{y}}{2^n}\right)\cos 2^n\pi\frac{x}{y}$$

$$+\sum_{n=\alpha+1}^{\infty}\lim_{\delta=0}\frac{\varphi(\pm\sin 2^n\pi\delta)}{(2^n)^2\delta}$$

where α is defined as on page 10. The character of this series will depend wholly upon the character of the last summation since by hypothesis $\varphi'(y)$ was to exist and have a definite value for $y \neq 0$, and in each term of the first series, $\sin 2^n\pi\frac{x}{y} \neq 0$. Hence, since each of the terms has a fixed value and there are a finite number of them, this part of our derivative

will have a definite value.

The dependence of the derivative of $f(x)$ in the points under discussion i.e. the rational points, upon the derivative of $\varphi(y)$ in the point $y=0$, may be made more clear by writing

$$\lim_{\delta=0}\left\{\frac{f(\frac{m}{q}+\delta)-f(\frac{m}{q})}{\delta}\right\} = \Pi \sum_{n=1}^{\alpha} \frac{\varphi'(\sin \lfloor n \Pi \frac{m}{q})}{\lfloor n} \cos \lfloor n \Pi \frac{m}{q}$$

$$+ \lim_{\delta=0} \sum_{n=\alpha+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \Pi \delta)}{(\lfloor n)^2 \delta}$$

$$= g + \lim_{\delta=0} \sum_{n=\alpha+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \Pi \delta)}{(\lfloor n)^2 \delta}$$

From this last expression it may be seen that the whole character of the derivative as to whether it is definite or indefinite in the point $x=\frac{m}{q}$, depends upon whether $\varphi'(y)$, or rather $\frac{\varphi(\delta)}{\delta}$, is fixed or indeterminate in the point $y=0$, or as $\delta=0$.

Three cases, similar to those under discontinuity, may arise here also, namely;

(a)

$$\lim_{\delta=0}\left\{\frac{f(\frac{m}{q}+\delta)-f(\frac{m}{q})}{\delta}\right\} = g + \lim_{\delta=0} \sum_{n=\alpha+1}^{\infty} \frac{\varphi(\pm \sin \lfloor n \Pi \delta)}{(\lfloor n)^2 \delta}$$

$$\lim_{\delta=0}\left\{\frac{f(x_0-\delta)-f(x_0)}{\delta}\right\} = g + \lim_{\delta=0}\sum_{n=\alpha+1}^{\infty}\frac{\varphi(-\sin \lfloor n\,\pi\,\delta)}{(\lfloor n)^2\,\delta}$$

(A).

$$\lim_{\delta=0}\left\{\frac{f(x_0+\delta)-f(x_0)}{\delta}\right\} = g + \frac{1}{(\lfloor\alpha+1)^2}\left\{\lim_{\delta=0}\frac{\varphi(-\sin\lfloor\alpha+1\,\pi\,\delta)}{\delta}\right\}$$

$$+\lim_{\delta=0}\sum_{n=\alpha+1}^{\infty}\frac{\varphi(\sin\lfloor n\,\pi\,\delta)}{(\lfloor n)^2\,\delta}$$

$$\lim_{\delta=0}\left\{\frac{f(x_0-\delta)-f(x_0)}{\delta}\right\} = g + \frac{1}{(\lfloor\alpha+1)^2}\left\{\lim_{\delta=0}\frac{\varphi(-\sin\lfloor\alpha+1\,\pi\,\delta)}{\delta}\right\}$$

$$+\lim_{\delta=0}\sum_{n=\alpha+1}^{\infty}\frac{\varphi(-\sin\lfloor n\,\pi\,\delta)}{(\lfloor n)^2\,\delta}$$

(C).

$$\lim_{\delta=0}\left\{\frac{f(x_0+\delta)-f(x_0)}{\delta}\right\} = g + \frac{1}{(\lfloor\alpha+1)^2}\left\{\lim_{\delta=0}\frac{\varphi(-\sin\lfloor\alpha+1\,\pi\,\delta)}{\delta}\right\}$$

$$+\frac{1}{(\lfloor\alpha+2)^2}\left\{\lim_{\delta=0}\frac{\varphi(-\sin\lfloor\alpha+2\,\pi\,\delta)}{\delta}\right\}$$

$$+\lim_{\delta=0}\sum_{n=\alpha+3}^{\infty}\frac{\varphi(\sin\lfloor n\,\pi\,\delta)}{(\lfloor n)^2\,\delta}$$

$$\lim_{\delta=0}\left\{\frac{f(x_0-\delta)-f(x_0)}{\delta}\right\} = g + \frac{1}{(\lfloor\alpha+1)^2}\left\{\lim_{\delta=0}\frac{\varphi(\sin\lfloor\alpha+1\,\pi\,\delta)}{\delta}\right\}$$

$$+\frac{1}{(\lfloor\alpha+2)^2}\left\{\lim_{\delta=0}\frac{\varphi(\sin\lfloor\alpha+2\,\pi\,\delta)}{\delta}\right\}$$

$$+\lim_{\delta=0}\sum_{n=\alpha+3}^{\infty}\frac{\varphi(-\sin\lfloor n\,\pi\,\delta)}{(\lfloor n)^2\,\delta}$$

(A full explanation of the manner in which these three cases (a), (b) and (c)

arise may be found on pages 12 14.)

From these three sets of equations (a), (b), and (c) we may make the following deductions in regard to the behavior of $f(x)$ in the rational points according to the characteristics of $\varphi'(y)$ in the point $y=0$.

(1) Let $\varphi'(y)$ be defined as under (1), page 84, i. e. it is finite and determined in the point $y=0$.

From (a) we have

$$\lim_{\delta=0}\left\{\frac{f(\frac{h}{q}+\delta)-f(\frac{h}{q})}{\delta}\right\} = g + \lim_{\delta=0}\sum_{n=\alpha+1}^{\infty}\frac{\varphi(\sin \lfloor n \pi\delta)}{(\lfloor n)^2\delta}$$

$$= g + \sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n} \cdot (\pi)(A)$$

$$\lim_{\delta=0}\left\{\frac{f(\frac{h}{q}-\delta)-f(\frac{h}{q})}{\delta}\right\} = g + \lim_{\delta=0}\sum_{n=\alpha+1}^{\infty}\frac{\varphi(-\sin \lfloor n \pi\delta)}{(\lfloor n)^2\delta}$$

$$= g + \sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n} \cdot \pi \cdot A.$$

where A is the value of $\varphi(\delta)/\delta$ in the limit when $\delta = 0$.

From (b) we get

$$\lim_{\delta=0}\left\{\frac{f(\frac{h}{q}\pm\delta)-f(\frac{h}{q})}{\delta}\right\} = g + \frac{1}{(\lfloor \alpha+1)^2}\lim_{\delta=0}\frac{\varphi(-\sin \lfloor \alpha+1\ \pi\delta)}{\delta}$$

$$+ \lim_{\delta=0}\sum_{n=\alpha+2}^{\infty}\frac{\varphi(\sin \lfloor n \pi\delta)}{(\lfloor n)^2\delta}$$

$$= g + \frac{1}{\lfloor\alpha+1} \Pi \cdot A + \sum_{n=\alpha+2}^{\infty} \frac{1}{\lfloor n} \Pi \cdot A$$

Lastly, from (c), in like manner, we have

$$\lim_{\delta=0} \left\{ \frac{f(x+\delta) - f(x)}{\delta} \right\} = g + \left\{ \frac{1}{\lfloor\alpha+1} + \frac{1}{\lfloor\alpha+2} \right\} \Pi \cdot A \cdot + \sum_{n=\alpha+3}^{\infty} \frac{1}{\lfloor n} \Pi A$$

Therefore $f(x)$ has a fixed derivative for every rational point when $\varphi(y)$ has a fixed derivative at the point $y = 0$.

(2). Let us take $\varphi(y)$ now as defined under (2), page 84, i. e. $\varphi'(y)$ is infinite and of determined sign for $y = 0$. We have then from (a),

$$\lim_{\delta=0} \left\{ \frac{f(x \pm \delta) - f(x)}{\delta} \right\} = g + \frac{\varphi(\pm\delta)}{\delta} \Pi \sum_{n=\alpha+1}^{\infty} \frac{1}{\lfloor n}$$

$$= +\infty \quad \text{or} \quad -\infty$$

according to the sign of $\varphi'(y)$ in the point $y = 0$.

From (b) we have

$$\lim_{\delta=0} \left\{ \frac{f(x \pm \delta) - f(x)}{\delta} \right\} = g + \frac{\Pi}{\lfloor\alpha+1} \frac{\varphi(\mp\delta)}{\delta} + \frac{\varphi(\pm\delta)}{\delta} \Pi \sum_{n=\alpha+2}^{\infty} \frac{1}{\lfloor n}$$

$$= +\infty \quad \text{or} \quad -\infty$$

as under (a).

From (c) we get a similar expression with the same result.

Therefore, $f(x)$ has a derivative

which is infinite and of determined sign for every rational point in how $\varphi(y)$ has such a derivative in the point $y=0$.

(3) Let $\varphi(y)$ be defined as under (3), page 84, i.e. $\varphi'(y)$ is infinite and of undetermined sign for $y=0$.

From (a) we get

$$\lim_{\delta=0}\left\{\frac{f(x\pm\delta)-f(x)}{\delta}\right\} = g \pm \infty$$

From (b) we have again

$$\lim_{\delta=0}\left\{\frac{f(x\pm\delta)-f(x)}{\delta}\right\} = g \mp \infty \pm \infty$$

(c) gives the same result as (b). Thus in case $\varphi'(y)$ is infinite and of undetermined sign when $y=0$, in the points where k is even, (see page 12) $f(x)$ is also infinite and of undetermined sign, while in the points for which k is odd $f'(x)$ is wholly indeterminate.

(4) Let $\varphi(y)$ be defined as under (4), page 84, i.e. $\varphi'(y)$ is wholly indeterminate for $y=0$. From (a) we have

$$\lim_{\delta=0}\left\{\frac{f(x+\delta)-f(x)}{\delta}\right\} = g + ?$$

where the interrogation point is used

to show that there is some quantity, which is wholly indetermined to g. From (b) and (c) we get similar results. Therefore, when $\varphi'(y)$ is wholly indeterminate in the point $y = a$, $f(x)$ is wholly indeterminate in every rational point.

As an illustration of the method employed in this part of the section let us take the function given under (3), page 84, namely

$$\varphi(y) \equiv \log \sin y. \qquad \text{(See page 97)}$$

Then

$$\varphi'(y) \equiv \cot y.$$

Forming our new function we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\log \sin(\sin \lfloor n \pi x)}{(\lfloor n)^2}$$

Moreover, the sum of the derivatives of the individual terms of this series is

$$\pi \sum_{n=1}^{\infty}{}' \frac{\cos(\sin \lfloor n \pi x)}{\sin(\sin \lfloor n \pi x)} \cdot \frac{\cos \lfloor n \pi x}{\lfloor n}$$

$$= \pi \sum_{n=1}^{\infty}{}' \frac{\cot(\sin \lfloor n \pi x)}{\lfloor n} \cos \lfloor n \pi x.$$

For the irrational values of x, $\sin \lfloor n \pi x$ may be very near to zero in a large

number of terms, and this will make $\cot(\sin \lfloor m \pi x)$ very large, and we can say nothing in regard to the derivative of $f(x)$ on the irrational points.

For the rational points we have

$$f'\left(\frac{h}{g}\right) = \Pi \sum_{m=1}^{m=\alpha-1} \frac{\cot\left(\sin \lfloor m \Pi \frac{h}{g}\right)}{\lfloor m} \cos \lfloor m \Pi \frac{h}{g}$$

$$+ \Pi \sum_{n=\alpha}^{n=\infty} \frac{\cot(\sin k \Pi)}{\lfloor n} \cos k \Pi \qquad ; k = \frac{\lfloor m}{g} h$$

since $\cot x$ for $x=0$ approaches $+\infty$ thro' positive values of x, and $-\infty$ thro' negative values the last summation above may be either $+\infty$ or $-\infty$ or wholly indeterminate in the point $x = h/g$. Let us examine the terms in this series and find how $\cot(\sin k\Pi)$ behaves for positive and negative increments when k is even and when k is odd.

1) Let k be even. Then $\sin(k\Pi+\delta)$ is positive and $\sin(k\Pi-\delta)$ is negative, and $\cos(k\Pi+\delta)$ and $\cos(k\Pi-\delta)$ are both positive. Our second summation above then becomes $+\infty$ on the right and $-\infty$ on the left of the point $x = h/g$, since when the first k is even all the remaining k's

are even also.

2) Let the first k be odd. Then $\sin(k\pi+\delta)$ is negative, $\sin(k\pi-\delta)$ is positive, $\cos(k\pi+\delta)$ and $\cos(k\pi-\delta)$ are both negative and again we have $+\infty$ as the limit upon the right of the point $x = p/q$, and $-\infty$ upon the left.

Since the results are the same whether k is even or odd we have

$$f'(p/q) = \pi \sum_{n=1}^{\alpha-1} \frac{\cot\left(\sin \lfloor n \pi \frac{p}{q}\right)}{\lfloor n \rfloor} \cos \lfloor n \pi \frac{p}{q}$$

$+\infty$

as we approach p/q from the right, and

$$f'(p/q) = \pi \sum_{n=1}^{\alpha-1} \frac{\cot\left(\sin \lfloor n \pi \frac{p}{q}\right)}{\lfloor n \rfloor} \cos \lfloor n \pi \frac{p}{q}$$

$-\infty$

as we approach from the left. The first summation above is finite since $\cot x$ is finite for $x \neq 0$. Hence

$$f'(p/q) = \infty$$

to the right and

$$f'(p/q) = -\infty$$

to the left; i.e. $f(x)$ behaves at every rational point in the same manner that $\varphi(y)$ does at the point $y = 0$.

$\varphi(y)$.

Section VII

In this section we shall apply the method of condensation devised by Cantor to the same classes of functions which we discussed in the last section by another method. These classes are, (a) those which have everywhere except at the point $y=0$, a definite and fixed derivative and, (b), those which have a fixed and definite derivative outside of the point $y=0$, but in certain points, as y approaches indefinitely near to zero this derivative may become greater than any fixed number g.

(a)

Let us define $\varphi(y)$ now as a function of y, which, between the limits a and b is finite and continuous, and, except at the point $y=0$, has a fixed and finite derivative, which is less than some fixed number g, and also so that $\varphi(y)=0$ for $y=0$.

We now define $f(x)$ in the same manner as in section nam-

ely

$$f(X) = \sum_{n=1}^{\infty} C_n \varphi(X - w_n)$$

where C_n and w_n are defined as in the section referred to, and

$$a - X\, w_n - b.$$

For $X \neq w_n$, since $\varphi(y)$ has a definite and finite derivative for every point outside the point y - we have

$$f'(X) \leq \sum_{n=1}^{\infty} g C_n = g \sum_{n=1}^{\infty} C_n$$

where g is the upper limit of the derivative of $\varphi(y)$ in the given interval. Hence $f(x)$ has a fixed and finite derivative for every point which does not belong to the Menge of w's.

Let us now take $X = w_K$. We have then,

$$f(X) = \sum_{n=1}^{n=k-1} C_n \varphi(X - w_n) + C_k \varphi(0) + \sum_{n=k+1}^{\infty} C_n \varphi(X - w_n)$$

and therefore, differentiating term by term we get

$$f'(X) = C_k \varphi'(0) + \sum_{n=1}^{k-1} C_n \varphi'(X - w_n) + \sum_{n=k+1}^{\infty} C_n \varphi'(X - w_n)$$

The question now is whether this series actually represents the derivative of $f(X)$ in the given point. The two summations in this last series will have definite, finite values, but since by hypothesis

$\varphi'(0)$ is indeterminate (i.e. has no limit) then $f'(x)$ cannot exist at the given point. That is, if $\varphi(y)$ is a function of y, such that in the interval between a and b, $\varphi'(y)$ is always finite and less than some number g, except at the point $y=0$, and $\varphi'(0)$ is indeterminate, then

$$f(x) = \sum_{n=1}^{\infty} C_n \varphi(x - a_n)$$

will have a fixed and finite derivative in the points which do not belong to the Menge of a_n, while for all points which do belong to this Menge, it has no fixed derivative. Moreover it may be readily seen from the form of the last equation on the preceding page, that if $\varphi'(y)$ has a fixed limit upon both sides of the point $y=0$, $f(x)$ will have the same characteristic at the point $x = a_k$.

Example.

Given

$$\varphi(y) = y \sin \frac{1}{y}.$$ (See page 83)

This function, as we proved on page of this thesis, has no fixed deriv

alive at the point $y=c$, but at every other point the fixed derivative exists. forming our function $f(x)$ we have,

$$f(x) \equiv \sum_{n=1}^{n=\infty} C_n (x - w_n) \sin \frac{1}{(x-w_n)}$$

letting

$$C_n = \frac{1}{\lfloor n}$$

and w_n be any rational number, these numbers being arranged as on page , we have,

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{\lfloor n} (x - w_n) \sin \frac{1}{(x-w_n)}$$

$$\equiv (x-0) \sin \frac{1}{x-0} + \frac{1}{\lfloor 2} (x-1) \sin \frac{1}{x-1} + \frac{1}{\lfloor 3} (x-\tfrac{1}{2}) \sin \frac{1}{(x-\frac{1}{2})} + \cdots\cdots$$

Since the derivative of $y \sin \frac{1}{y}$ is fixed and finite for every value of y different from zero we may write the equation above

$$f(x) \equiv x \sin \frac{1}{x} + \frac{1}{\lfloor 2} (x-1) \sin \frac{1}{(x-1)} + \frac{1}{\lfloor 3} (x-\tfrac{1}{2}) \sin \frac{1}{x-\frac{1}{2}} + \cdots\cdots$$

and its derivative will be

$$f'(x) \equiv \frac{\cos \frac{1}{x}}{x} + \sin \frac{1}{x} + \frac{1}{\lfloor 2} \left(\frac{(x-1) \cos 1/x-1}{(x-1)^2} + \sin \frac{1}{x-1} \right)$$

$$+ \frac{1}{\lfloor 3} \left(\frac{(x-\frac{1}{2}) \cos 1/x-\frac{1}{2}}{(x-\frac{1}{2})^2} \sin \frac{1}{(x-\frac{1}{2})} \right) + \cdots\cdots$$

This expression represents a definite, fixed quantity for any value of x which does not belong to the menge $\frac{n}{n}$ rational

numbers n such that for every X in such does belong to this range we shall have some term of the form

$$\frac{1}{(n\pi)} \sin \frac{1}{x}$$

since the derivative of $x \sin \frac{1}{x}$ on the point $x = 0$ is $\sin \frac{1}{0}$.* This is indeterminate, and hence for any rational number $f'(x)$ is indeterminate. The amount of oscillation will depend upon the value of n in the term in which

$$(x - (1/n)) = 0$$

* See page 79.

(6).

$\varphi(y)$ is now taken as a function of y such that within any finite interval

$$a < y < b$$

it has the following characteristics:

$$\text{For } |y| \leq 1, \quad \varphi(y) < g_2$$
$$\text{For } |y| > 1, \quad \varphi(y)/y^\alpha < g_1$$

where g_1 and g_2 are fixed positive numbers and α is a positive integer. Moreover $\varphi'(y)$ is fixed and definite for every value of y different from zero, but as y approaches indefinitely near to zero, $\varphi'(y)$ becomes greater than any fixed value g but fulfills the following conditions, namely:-

$$\text{For } |y| \leq 1, \quad |y^\beta \varphi'(y)| < g''$$
$$\text{For } |y| > 1, \quad |\varphi'(y)| < y^\gamma g'''$$

where g'' and g''' are fixed positive numbers, and β and γ are positive integers. No conditions whatever are placed upon the behavior of $\varphi'(y)$ in the point $y = 0$.

$$f(x) \equiv \sum_{n=1}^{\infty} c_n \varphi(x - a_n)$$

will then be a uniformly convergent series within the limits

$$a - x\,u_n < b$$

since by our definition of $\psi(u)$ we have for this interval

$$f(x) = \sum_{n=1}^{\infty} c_n \psi(x - u_n) < |b_1|^q \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where b_1 is the numerically larger of the two numbers a and b, q is the larger of the two numbers g_1 and g_2, and $c_n = \frac{1}{n^s}$.

We wish now to let our menge of u's be the rational points in the given interval. In order to have a convenient method of handling them let us arrange them in groups according to the value of the sum of their numerators and denominators; i.e. all the rational numbers, h/k, for which $h+k$ is equal to a certain constant, constitute a group. The arrangement then is as follows;

$$-1, 1, -2, -\tfrac{1}{2}, \tfrac{1}{2}, 2, -3, -\tfrac{1}{3}, \tfrac{1}{3}, 3, -4, -\tfrac{2}{3}, -\tfrac{1}{4}, \tfrac{1}{4}, \ldots\ldots$$

We now designate the K^{th} number in this series a menge as u_K in our function $f(x)$. The subscript K of any number in this series is, then, larger than the absolute value of that number, and also greater than the denom-

inals of that number. Hence,

$$\left|\frac{p_q}{q} - \frac{p_\nu}{\nu}\right| \geqq \frac{1}{q\nu} \geqq \frac{1}{\lambda n}$$

where p_q/q and p_ν/ν are any numbers of this merge and λ and n are their numbers in the merge. It follows then from the inequalities above that

$$|w_\lambda - w_n| > \frac{1}{\lambda n}$$

or

$$\lambda n > \frac{1}{|w_\lambda - w_n|}$$

Now let us examine the derivative of $f(x)$ in these rational points. For those terms in which

$$|w_\lambda - w_n| \leqq 1$$

we have

$$\varphi'(w_\lambda - w_n) < g'' |w_\lambda - w_n|^{-\beta}$$

from our definition of $\varphi'(1)$. But we have seen above that

$$\lambda n > \frac{1}{|w_\lambda - w_n|}.$$

Hence we may write

$$\varphi'(w_\lambda - w_n) < g'' \lambda^\beta n^\beta$$

when

$$|w_\lambda - w_n| \leqq 1.$$

For the terms in which

$$|w_\lambda - w_n| > 1$$

we have

$$\varphi'(w_k - w_n) < g''' |w_k - w_n|^\nu$$

But it is easily seen that, since

$$\lambda > |w_k|$$

$$|w_k - w_n| < \lambda + n < 2\lambda n$$

Therefore

$$\varphi'(w_k - w_n) < g''' 2^\nu \lambda^\nu n^\nu$$

when

$$|w_k - w_n| > 1 .$$

Now if we let g' be the greater of the two numbers g'' and g''', and x the greater of the two numbers s and y, we may write for any term of our whole series

$$\varphi'(w_k - w_n) \leqq 2^x g' \lambda^x n^x .$$

The series of these derivatives then will fulfill the condition

$$\sum_{n=1}^{\infty} C_n \varphi'(w_k - w_n) \leqq 2^x g' \lambda^x \sum_{n=1}^{\infty} C_n n^x$$

$$\leqq 2^x g' \lambda^x \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where

$$C_n = n^{-s-x}$$

Since the value of this expression depends upon the value of λ, which has no upper limit, this series may not be uniformly convergent and we cannot yet say that the sum of

the derivatives of the individual terms is equal to the derivative of their sum, but must apply some other test in order to prove this.

If, now, we can prove that for the two series

$$\sum_{n=1}^{\infty} c_n \varphi(u_k - u_n)$$

and

$$\sum_{n=1}^{\infty} c_n \varphi'(u_k - u_n)$$

the term in which $n = k$ being omitted, the three expressions

$$\sum_{n=1}^{m} c_n \left\{ \frac{\varphi(u_k - u_n + \delta) - \varphi(u_k - u_n)}{\delta} - \varphi'(u_k - u_n) \right\}$$

$$\frac{R_m(u'_k + \delta)}{\delta}$$

and

$$\frac{R_m(u'_k)}{\delta}$$

are each less than some arbitrarily chosen small positive quantity σ, then we know that the second series is the derivative of the first. *

For convenience now let

$$D_n \equiv \left\{ \frac{\varphi(u_k - u_n + \delta) - \varphi(u_k - u_n)}{\delta} - \varphi'(u_k - u_n) \right\}$$

Then

$$\sum_{n=1}^{\infty}{}' c_n D_n \equiv \sum_{n=1}^{\infty}{}' c_n \left\{ \frac{\varphi(u_k - u_n + \delta) - \varphi(u_k - u_n)}{\delta} - \varphi'(u_k - u_n) \right\}$$

The accent mark is used here and will be

* See Dini page 151, § 103.

used over the summation sign, to indicate that the term in which n λ is to be omitted.

Let K and L be two numbers such that

$$K < L < \lambda.$$

We can break up our summation

$$\sum_{n=1}^{\infty} C_n D_n$$

into three parts, namely;

$$\sum_{n=1}^{K}{}' C_n D_n \quad , \quad \sum_{n=K+1}^{L}{}' C_n D_n \quad , \quad \sum_{n=L+1}^{\infty}{}' C_n D_n$$

Let us examine these summations separately, taking them in their order.

$$\sum_{n=1}^{K}{}' C_n D_n = \sum_{n=1}^{K}{}' C_n \left\{ \frac{\varphi(w_K - w_n + \delta) - \varphi(w_K - w_n)}{\delta} - \varphi(w_K - w_n) \right\}$$

Let us restrict δ now by the condition,

$$|\delta| < \frac{1}{2K n}$$

Then none of the terms in this summation will have an argument, $(w_K + \delta - w_K)$, which is equal to zero, since, as we have proven before

$$|w_K - w_n| > \frac{1}{K n}$$

and

$$|\delta| < \frac{1}{2K n}$$

Therefore this series becomes very small

as δ approaches zero, for by definition $\varphi(u)$ has a fixed derivative for every point $u \neq 0$. We have therefore for this part of our series

$$\sum_{n=1}^{k-1}{}' C_n D_n = \xi \sigma$$

where

$$|\xi| < 1.$$

The second part of our series is

$$\sum_{n=k+1}^{\infty}{}' C_n D_n = \sum_{n=k+1}^{l}{}' C_n \left\{ \frac{\varphi(w_k - w_n + \delta) - \varphi(w_k - w_n)}{\delta} - \varphi'(w_k - w_n) \right\}$$

Since $\varphi(u)$ has a fixed derivative for $u \neq 0$ and, as we proved on the preceding page, the arguments of the functions involved in this expression are different from zero, we may write:-

$$\left| \frac{\varphi(w_k - w_n + \delta) - \varphi(w_k - w_n)}{\delta} \right| = \left| \varphi'(w_k - w_n) + \theta_n \delta_1 \right|$$

where θ_n is some number between 0 and 1. From our work on page we see that

$$\varphi'(w_k - w_n) + \theta_n \delta < 2^{\chi} g' \lambda^{\chi} n^{-s}$$

Then since

$$\left| \frac{\varphi(w_k - w_n + \delta) - \varphi(w_k - w_n)}{\delta} \right| = \left| \varphi'(w_k - w_n) + \theta_n \delta_1 \right|$$

we may write

$$\sum_{n=k+1}^{l}{}' C_n D_n < 2 \cdot 2^{\chi} g' \lambda^{\chi} \sum_{n=k+1}^{l} \frac{1}{n^s}$$

We now take up the last part of our series, namely

$$\sum_{n=l+1}^{\infty} C_n D_n \equiv \sum_{n=l+1}^{\infty} C_n \left\{ \frac{\varphi(w_K - w_n + \delta) - \varphi(w_K - w_n)}{\delta} - \varphi'(w_K - w_n) \right\}$$

Separating this into its parts we have

$$\sum_{n=l+1}^{\infty} C_n D_n < \frac{1}{|\delta|} \left| Re(w_K + \delta) \right| + \frac{1}{|\delta|} \left| Re(w_K) \right| + \left| R'e(w_K) \right|$$

But from page 104 we see that

$$Re(w_K + \delta) \leq |b_1|^{\alpha} g \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

and

$$(Re(w_K) \leq |b_1|^{\alpha} g \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

From page 106 again

$$R'e(w_K) < 2^{\chi} g' \lambda^{\chi} \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

We have then

$$\sum_{n=l+1}^{\infty} C_n D_n < \frac{2g}{|\delta|} |b_1|^{\alpha} \sum_{n=l+1}^{\infty} \frac{1}{n^s} + 2^{\chi} g' \lambda^{\chi} \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

Our whole summation then becomes,

$$\sum_{n=1}^{\infty} C_n D_n = \xi \sigma + \theta_1 2^{\chi} g' \lambda^{\chi} \sum_{n=K+1}^{l} \frac{1}{n^s} + \theta_2 \frac{2g}{\delta} |b_1|^{\alpha} \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

$$+ \theta_3 2^{\chi} g' \lambda^{\chi} \sum_{n=l+1}^{\infty} \frac{1}{n^s}$$

where ξ, θ_1, θ_2, and θ_3 are proper fractions. For convenience let us put G_1, G_2 and G_3 for the constant parts of the terms on the right of the sign of equality. We have then,-

$$\sum_{n=1}^{\infty} G_n D_n = \xi\sigma + A_1 g_1 \sum_{n=k+1}^{\infty} \frac{1}{n^3} + A_2 g_2 \sum_{n=l+1}^{\infty} \frac{1}{n^3}$$

$$+ A_3 g_3 \sum_{n=l+1}^{\infty} \frac{1}{n^3}$$

We can, by increasing k and l sufficiently, make all these terms approach zero, unless it be the third, in which we have δ in the denominator. It is necessary, then, for us to examine this term more carefully.

We have already restricted $|\delta|$ by the condition

$$|\delta| < \frac{1}{2\lambda n}$$

or in the case in question

$$|\delta| < \frac{1}{2\lambda l}$$

Then

$$l < \frac{1}{2\lambda|\delta|}$$

If we take l as the next smaller number below $\frac{1}{2\lambda|\delta|}$ we have

$$1 + l > \frac{1}{2\lambda|\delta|}$$

or

$$2\lambda|\delta|(1+l) > 1$$

This may be written

$$2\lambda|\delta| + 2\lambda l|\delta| > 1$$

Then

$$2\lambda l|\delta| > 1 - 2\lambda|\delta|$$

and

$$l > \frac{1 - 2\lambda|\delta|}{2\lambda|\delta|}$$

Raising both sides to the $(s-1)^{st}$ power

$$\ell^{s-1} > \left\{1 - \frac{2\lambda|\delta|}{2\lambda|\delta|}\right\}^{s-1}$$

$$> |\delta|^{1-s}\left\{\frac{1}{2\lambda} - |\delta|\right\}$$

Multiplying thro by $|\delta|$

$$|\delta|\,\ell^{s-1} > |\delta|^{2-s}\left\{\frac{1}{2\lambda} - |\delta|\right\}$$

The expression on the right of the inequality sign grows larger than an assignable quantity as δ approaches zero, when $s > 2$. Hence $|\delta|\ell^{s-1}$ approaches zero as δ approaches zero. Then, since

$$\sum_{n=\ell+1}^{\infty} \frac{1}{n^s} < \frac{1}{\ell^{s-1}}$$

all of the terms in our equation

$$\sum_{n=1}^{\infty}{}' C_n D_n = \xi\tau + A_1 G_1 \sum_{n=K+1}^{\ell}{}' \frac{1}{n^s} + A_2 \frac{G_2}{\delta} \sum_{n=\ell+1}^{\infty}{}' \frac{1}{n^s} + A_3 G_3 \sum_{n=\ell+1}^{\infty}{}' \frac{1}{n^s}$$

have the lower limit zero and we may take the derivatives of the terms in the summation

$$\sum_{n=1}^{\infty}{}' C_n \varphi(u_K - u_n)$$

term by term and their sum will be equal to the derivative of the sum of the terms in the series. We may write then

$$\frac{f(x+\delta) - f(x)}{\delta} = \sum_{n=1}^{\infty}{}' \frac{1}{n^s} \psi'(u_K - u_n) + C_1 \frac{\varphi(\delta)}{\delta} + \eta$$

where η approaches zero as δ approaches

zero. The behavior of $f'(x)$, then, in the rational points is wholly dependent upon the behavior of $\varphi'(y)$ in the point $y=0$. This may be seen more clearly if we put a constant in the place of the summation; i.e

$$C = \sum_{n=1}^{n=\infty} \frac{\varphi'(u_k - u_n)}{n^s}$$

which we may do, since $\varphi(y)$ has a fixed derivative for every value of y different from zero. We have then

$$\frac{1}{\delta=0}\left\{\frac{f(x+\delta) - f(x)}{\delta}\right\} = C + \frac{1}{\delta=0}\frac{1}{x^s}\cdot\frac{\varphi(\delta)}{\delta}$$

$f(x)$, then will have the same characteristics, as regards its derivative, in the rational points that $\varphi(y)$ has in the point $y=0$.

Example.

Given;

$$\varphi(y) \equiv y\cos 1/y$$ (See page 116).

Taking the derivative we have

$$\varphi'(y) \equiv \cos 1/y - \frac{y \sin 1/y}{-y^2}$$

$$= \cos 1/y + \frac{\sin 1/y}{y}$$

As y approaches indefinitely near to zero $\cos 1/y$ will oscillate between -1 and $+1$, as will also $\sin 1/y$, but the denominator of the second becomes very small, so that

$$\frac{2 \sin 1/y}{y}$$

will become, in certain points, larger than any given finite number q. But if we multiply this by y we get

$$y \cdot \varphi'(y) \equiv y \cos 1/y + \sin 1/y$$

which is always less than 2 for every value of y less than 1. Moreover,

$$\varphi'(y) \equiv \cos 1/y + \frac{\sin 1/y}{y}$$

cannot be greater than 2 for any value of y greater than 1. Therefore this function obeys all the conditions imposed upon $\varphi(y)$ on page 103. Letting

$$c_n = 1/n^2$$

and forming $f(x)$ we have

$$f(x) \equiv \sum_{n=1}^{\infty} (x - a_n) \cos \left(\frac{1}{x - a_n} \right) \cdot \frac{1}{n^2}$$

where ω_n represents a rational number. Taking the derivative, we have,

$$f'(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ \cos^2\left(\frac{1}{x-\omega_n}\right) + \frac{\sin\left(\frac{1}{x-\omega_n}\right)}{(x-\omega_n)} \right\}$$

Since ω_n is a rational number, $f'(x)$ will be fixed for every irrational value of x. For any rational point

$$x = \omega_k$$

we have

$$f'(\omega_k) \equiv \sum_{n=1}^{k-1} \frac{1}{n^2} \left\{ \cos^2\left(\frac{1}{\omega_k-\omega_n}\right) + \frac{\sin\left(\frac{1}{\omega_k-\omega_n}\right)}{(\omega_k-\omega_n)} \right\}$$

$$+ \sum_{n=k+1}^{\infty} \frac{1}{n^2} \left\{ \cos^2\left(\frac{1}{\omega_k-\omega_n}\right) + \frac{\sin\left(\frac{1}{\omega_k-\omega_n}\right)}{(\omega_k-\omega_n)} \right\}$$

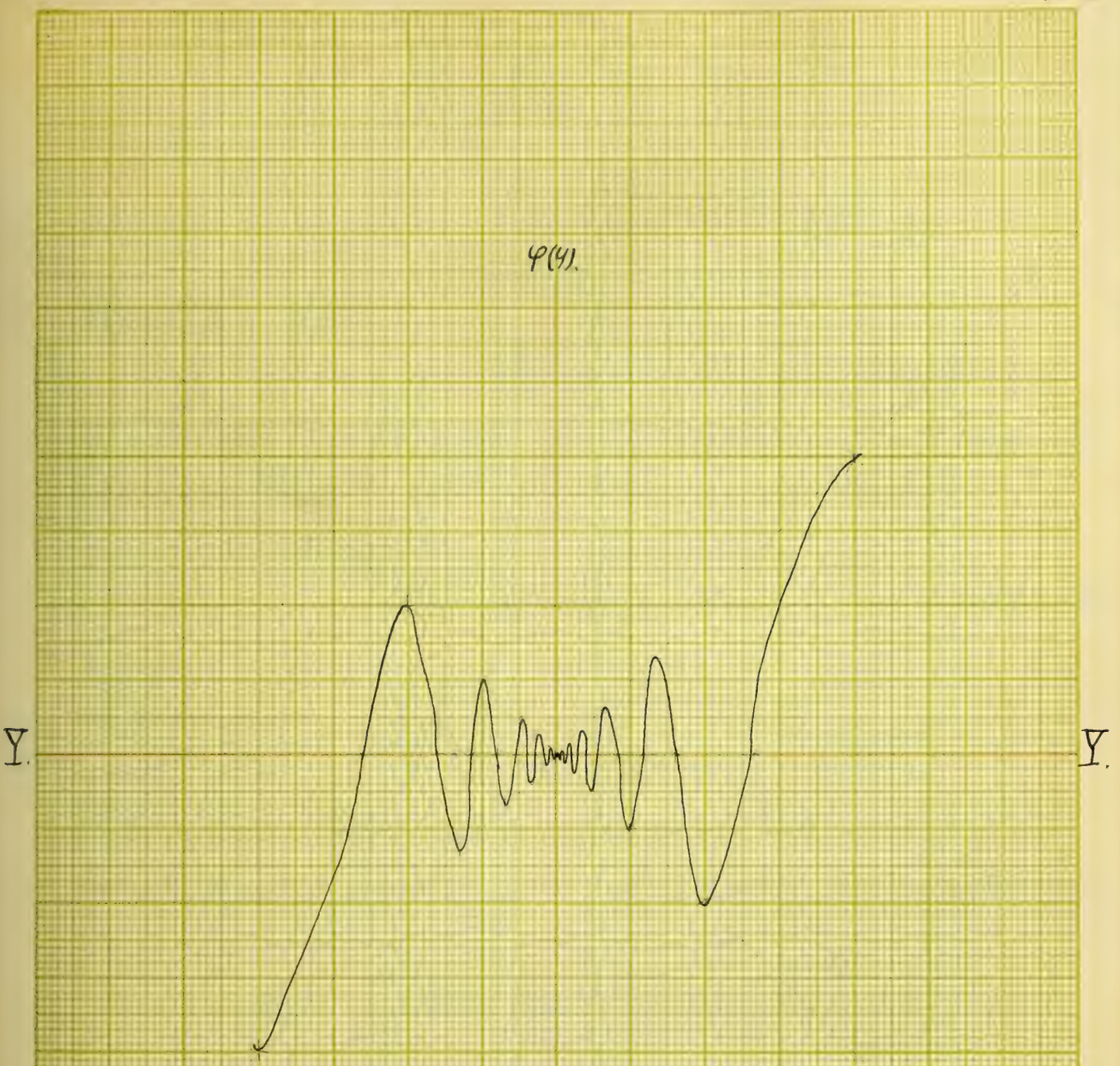
$$+ \frac{1}{k^2} \lim_{\delta=0} \frac{\left\{(\omega_k-(\omega_n\pm\delta))\left(\sin\left(\frac{1}{\omega_k-\omega_n\pm\delta}\right)-(\omega_k-\omega_n)\sin\left(\frac{1}{\omega_k-\omega_n}\right)\right)\right\}}{\delta}$$

$$= \sum_{n=1}^{k-1} \frac{1}{n^2} \left\{ \cos^2\left(\frac{1}{\omega_k-\omega_n}\right) + \frac{\sin\left(\frac{1}{\omega_k-\omega_n}\right)}{\omega_k-\omega_n} \right\}$$

$$+ \sum_{n=k+1}^{\infty} \frac{1}{n^2} \left\{ \cos^2\left(\frac{1}{\omega_k-\omega_n}\right) + \frac{\sin\left(\frac{1}{\omega_k-\omega_n}\right)}{\omega_k-\omega_n} \right\}$$

$$+ \frac{1}{k^2} \lim_{\delta=0} \sin\tfrac{1}{\delta}$$

This last term is indeterminate when $\delta = 0$, therefore $f(x)$ has no fixed derivative at any rational point.



Rough Sketch
showing the behavior
of the function

$\varphi(y) = y \cos 1/y$

in the neighborhood
of the point
$y = 0$.

Scale- 4 Squares = 1 Unit.

Chapter III.

Problems.

Section VIII

This section will be devoted to the solution of problems illustrating the method set forth in section I.

(1.) Given;

$$\varphi(y) \equiv \lim_{m=\infty} \left\{ \frac{(1+y^2)^m - 1}{(1+y^2)^m + 1} \right\}$$ (See page 119)

Then for $y \neq 0$, $\varphi(y) = 1$.
and for $y = 0$, $\varphi(y) = 0$.
Therefore $\varphi(y)$ has a discontinuity of the first kind upon both sides of the point $y = 0$. Forming $f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\lfloor n} \lim_{m=\infty} \left\{ \frac{(1+\sin^2 \lfloor n \, \pi x)^m - 1}{(1+\sin^2 \lfloor n \, \pi x)^m + 1} \right\}$$

Since $\sin \lfloor n \, \pi x \neq 0$ for all irrational values of x and $\sin \lfloor n \, \pi x = 0$ for rational values, as $\lfloor n$ grows large we have

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\lfloor n} = (e-1)$$

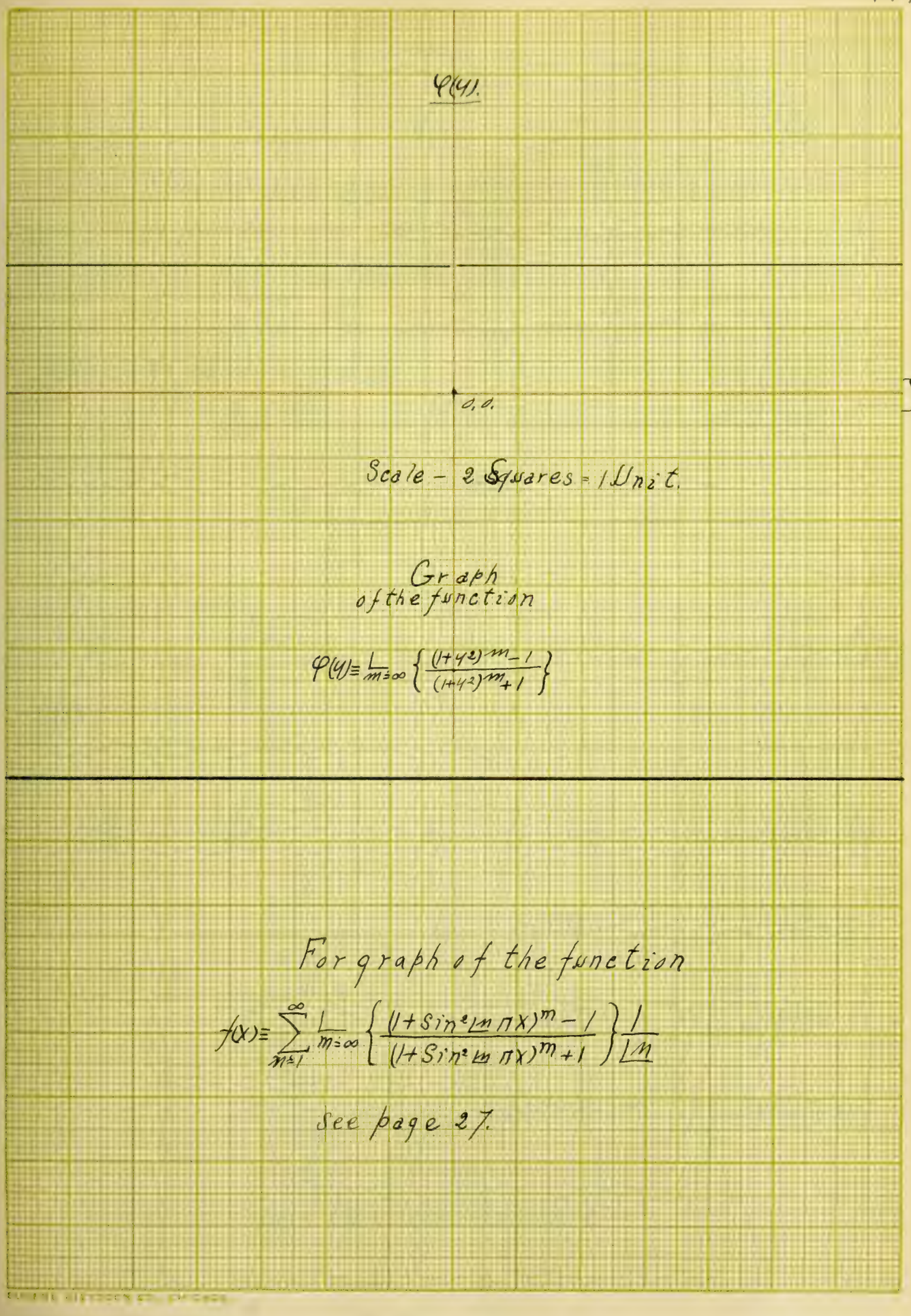
for all irrational values of x, and

$$f(x) = \sum_{n=1}^{\alpha-1} \frac{1}{|n|} + O$$

where α is the smallest number for which

$$f(x) = K\,g$$

makes K an integer. (Our function $f(x)$ then has a discontinuity of the first kind upon both sides of the point $x = n/g$. (For plot of both of these functions, $\psi(y)$ and $f(x)$ see next pages).



Scale – 2 Squares = 1 Unit.

Graph
of the function

$$\varphi(y) \equiv \lfloor_{m=\infty} \left\{ \frac{(1+y^2)^m - 1}{(1+y^2)^m + 1} \right\}$$

For graph of the function

$$f(x) \equiv \sum_{m=1}^{\infty} \lfloor_{m=\infty} \left\{ \frac{(1+\sin^{2\lfloor m} \pi x)^m - 1}{(1+\sin^{2\lfloor m} \pi x)^m + 1} \right\} \frac{1}{\lfloor m}$$

See page 27.

(2) Given;

$$\varphi(y) = \frac{1}{1+e^{1/y}} \qquad \text{(See page 123)}$$

where we define $\varphi(y) = 0$ for $y = 0$. $\varphi(y)$ is then continuous right, but has a discontinuity of the first kind upon the left of the point $y = 0$. Moreover $\varphi(y)$ is continuous for every value of y different from zero.

Forming $f(x)$ now we have;

$$f(x) \equiv \sum_{m=1}^{\infty} \frac{1}{1+e^{\pm \sin m\pi x}} \cdot \frac{1}{m}$$

Since $\varphi(y)$ is continuous for every y different from zero, $f(x)$ is continuous for irrational values of x. For the rational values we have,

$$f\left(\frac{k}{q} \pm \delta\right) - f\left(\frac{k}{q}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \frac{1}{1+e^{\pm \sin m\pi\left(\frac{k}{q} \pm \delta\right)}} - \frac{1}{1+e^{\pm \sin m\pi \frac{k}{q}}} \right\}$$

$$= \sum_{m=1}^{\alpha-1} \frac{1}{m} \left\{ \frac{1}{1+e^{\pm \sin m\pi\left(\frac{k}{q} \pm \delta\right)}} - \frac{1}{1+e^{\pm \sin m\pi \frac{k}{q}}} \right\}$$

$$+ \sum_{m=\alpha}^{\infty} \frac{1}{m} \left\{ \frac{1}{1+e^{\pm \sin(k\pi \pm m\pi\delta)}} - \frac{1}{1+e^{\pm \sin k\pi}} \right\}$$

$$= \sum_{m=1}^{\alpha-1} \frac{1}{m} \left\{ \frac{1}{1+e^{\pm \sin m\pi\left(\frac{k}{q} \pm \delta\right)}} - \frac{1}{1+e^{\pm \sin m\pi \frac{k}{q}}} \right\}$$

$$+ \sum_{m=\alpha}^{\infty} \frac{1}{m} \left\{ \frac{1}{1+e^{\pm \sin m\pi\delta}} \right\}$$

Since $\sin m\pi \frac{p}{q} \neq 0$ in each of the terms in the first summation we may leave them out in an examination of $f(x)$ at this point and we have, when,

$$p \pm m = 2k'q,$$

$$f\left(\frac{p}{q}+\delta\right) - f\left(\frac{p}{q}\right) = \sum_{n=\alpha}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\pm i m\pi \frac{p}{q}}} \right\}$$

The limit of this expression, as δ approaches zero is zero. Hence $f(x)$ is continuous upon the right of this point. On the other side

$$f\left(\frac{p}{q}-\delta\right) - f\left(\frac{p}{q}\right) = \sum_{n=\alpha}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\mp i m\pi\delta}} \right\}$$

The limit of this expression is

$$\sum_{n=\alpha}^{\infty} \frac{1}{n^m}$$

as δ approaches zero. Therefore $f(x)$ has a discontinuity of the first kind upon the left of the point in question.

When

$$p \pm m = (2k'+1)q$$

we have,

$$f\left(\frac{p}{q}+\delta\right) - f\left(\frac{p}{q}\right) = \frac{1}{\alpha^m} \frac{1}{1+e^{\mp i m\pi\delta}} + \sum_{n=\alpha+1}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\pm i m\pi\delta}} \right\}$$

The limit of this expression as δ approaches zero is

$$\frac{1}{k\pi}.$$

Therefore $f(x)$ has a discontinuity of the first kind upon the right of this point. On the other side we have

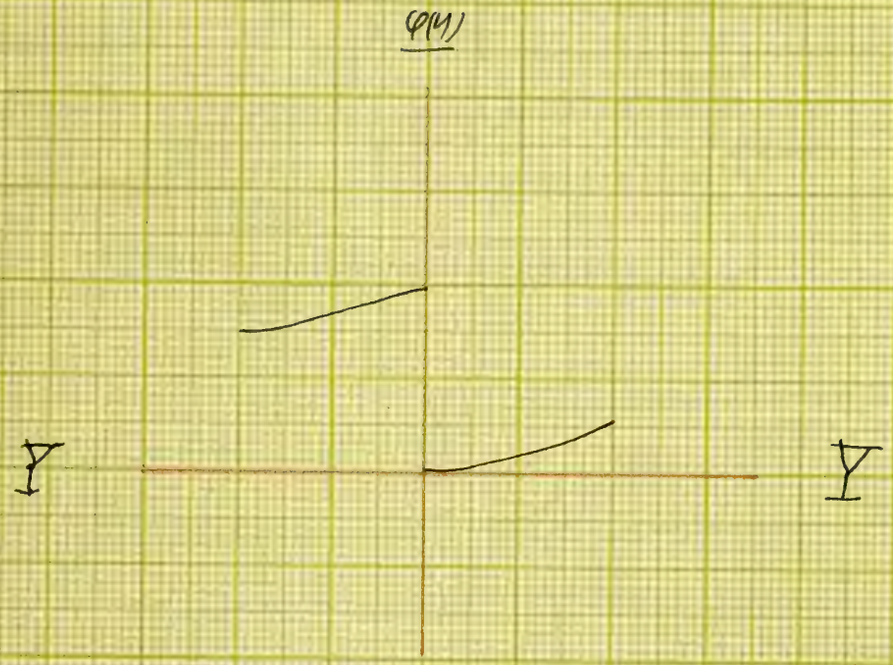
$$f\left(\tfrac{h}{g}-\delta\right)-f\left(\tfrac{h}{g}\right)=\frac{1}{k\pi}\frac{1}{1+e^{\frac{2\pi i n}{k}\delta}}+\sum_{n=\alpha+1}^{\infty}\frac{1}{n\pi}\frac{1}{1+e^{-\frac{2\pi i n}{k}\delta}}\Bigg\}$$

The limit of this expression, as δ approaches zero is

$$\sum_{n=\alpha+1}^{\infty}\frac{1}{2n\pi}.$$

Hence $f(x)$ has a discontinuity of the first kind upon the left of the point in question.

$f(x)$, then, is continuous at every irrational point; at the points $x=\frac{h}{g}$, for which k is even $f(x)$ is continuous on the right but has a discontinuity of the first kind upon the left; at the points $x=\frac{h}{g}$ for which k is odd $f(x)$ has a discontinuity of the first kind upon both sides of the point in question.



Sketch
of the function

$$\varphi(y) = \frac{1}{1+e^{1/y}}$$

near the point
$y = 0$

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(3) Given:

$\varphi(y) = \sin 1/y$ (See pag 131)

$\sin 1/y$ oscillates between -1 and $+1$, as y approaches zero, and becomes indeterminate in the point $y=0$, Therefore $\varphi(y)$ has a discontinuity of the second kind at the point $y=0$. (See page 131 for rough plot of this function.)

Forming $f(x)$ we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{\sin n! \pi x}$$

$\sin n! \pi x \neq 0$ for all irrational values of x, and $\sin 1/y$ is fixed, less than one, and continuous for all values of different from zero. Therefore $f(x)$ is fixed, finite, and continuous when x is irrational.

For the rational values of x let us apply the ordinary test for discontinuity, namely;

$$f(q/p \pm \delta) - f(q/p) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \left(\sin \frac{1}{\sin n! \pi (q/p \pm \delta)} - \sin \frac{1}{\sin n! \pi q/p} \right)$$

$$= \sum_{n=1}^{\alpha-1} \frac{1}{n} \left(\sin \frac{1}{\sin n! \pi (q/p \pm \delta)} - \sin \frac{1}{\sin n! \pi q/p} \right)$$

$$+ \sum_{n=\alpha}^{\infty} \frac{1}{n} \left(\sin \frac{1}{(\pm \sin n! \pi \delta)} - \sin \frac{1}{\sin 0} \right)$$

$\sin \frac{1}{\sin\delta}$ is wholly indeterminate between the limits -1 and $+1$, and $\sin \frac{1}{\sin\delta}$ oscillates as δ approaches zero. Therefore $f(x)$ has also a discontinuity of the second kind at the point $x = \frac{h}{q}$, i.e. at every rational point.

(4). Given;

$$\varphi(y) = y + \sin 1/y + \sin|1/y| \quad \text{(See page 131)}$$

$\varphi(y)$ now has a discontinuity of the second kind upon the right of the point $y=0$, and is continuous upon the left of that point if we define $\varphi(y)=0$ for $y=0$, which we are at liberty to do. This may be made more clear by writing the function in a form which shows how the absolute value of y in the last term affects the value of the function. Since

$$|1/y| = 1/y$$

when y is positive, for all positive values of y we may write

$$\varphi(y) \equiv y + \sin 1/y + \sin 1/y$$
$$= y + 2\sin 1/y$$

Since $\sin 1/y$ oscillates as y approaches zero, this function $\varphi(y)$ must oscillate as y approaches zero thro positive values. Again

$$|1/y| = -(1/y)$$

when y is negative. Moreover

$$\sin 1/y = -\sin 1/-y$$

Then for negative values of y our function becomes;

$$\varphi(-y) \equiv -y + \sin 1/-y + \sin|1/-y|$$
$$= -y - \sin 1/y + \sin 1/y$$
$$= -y$$

which is simply one of the 45° lines and is a continuous function of y.

Forming $f(x)$ we have,

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{\lfloor n} \left(\sin \lfloor n \pi x + \sin \frac{1}{\sin \lfloor n \pi x} + \sin \left| \frac{1}{\sin \lfloor n \pi x} \right| \right)$$

$\varphi(y)$ is a continuous function of y for $y \neq 0$, and moreover is always less than some given number g within the interval

$$-1 \leqq y \leqq +1$$

hence $f(x)$ is continuous for all irrational values of x. For the rational values we must apply our usual test, namely

$$f(\tfrac{n}{q} \pm \delta) - f(\tfrac{n}{q}) \equiv \sum_{n=1}^{\alpha-1} \frac{1}{\lfloor n} \left\{ \sin \lfloor n \pi (\tfrac{n}{q} \pm \delta) - \sin \lfloor n \pi \tfrac{n}{q} \right.$$

$$+ \sin \frac{1}{\sin \lfloor n \pi (\tfrac{n}{q} \pm \delta)} - \sin \frac{1}{\sin \lfloor n \pi \tfrac{n}{q}} + \sin \left| \frac{1}{\sin n \pi (\tfrac{n}{q} \pm \delta)} \right|$$

$$\left. - \sin \left| \frac{1}{\sin \lfloor n \pi \tfrac{n}{q}} \right| \right\} + \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \left\{ \sin \lfloor n \pi (\tfrac{n}{q} \pm \delta) - \sin \lfloor n \pi \tfrac{n}{q} \right.$$

$$+ \sin \frac{1}{\sin \lfloor n \pi (\tfrac{n}{q} \pm \delta)} - \sin \frac{1}{\sin \lfloor n \pi \tfrac{n}{q}} + \sin \left| \frac{1}{\sin \lfloor n \pi (\tfrac{n}{q} \pm \delta)} \right|$$

$$\left. + \sin \left| \frac{1}{\sin \lfloor n \cdot \tfrac{n}{q} \pi} \right| \right.$$

since in the first summation $\sin \lfloor n \pi \frac{m}{q}$ is different from zero, all those terms become zero and we have, when the first k is even (see page 12)

(6).

$$f(\tfrac{m}{q}+\delta) - f(\tfrac{m}{q}) = \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \{ \sin \lfloor n \pi \delta + \sin \frac{1}{\sin \lfloor n \pi \delta} + \sin \left| \frac{1}{\sin \lfloor n \pi \delta} \right| \}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \{ \sin \lfloor n \pi \delta + 2 \sin \left| \frac{1}{\sin \lfloor n \pi \delta} \right| \}$$

which oscillates as δ approaches zero, hence at this point $f(x)$ has a discontinuity of the second kind upon the right. Upon the left we have;

$$f(\tfrac{m}{q}-\delta) - f(\tfrac{m}{q}) = \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \{ -\sin \lfloor n \pi \delta + \sin \frac{1}{-\sin \lfloor n \pi \delta} + \sin \left| \frac{1}{-\sin \lfloor n \pi \delta} \right| \}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \{ -\sin \lfloor n \pi \delta - \sin \frac{1}{\sin \lfloor n \pi \delta} + \sin \frac{1}{\sin \lfloor n \pi \delta} \}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \{ -\sin \lfloor n \pi \delta \}.$$

which can be made less than some arbitrarily chosen small positive quantity σ for some δ and every δ

smaller than that, Therefore $f(x)$ is continuous on the left of the point in question.

When k is odd we have.

(2).

$$f\left(\tfrac{n}{q}+\delta\right)-f\left(\tfrac{n}{q}\right)=\frac{1}{\lfloor\alpha}\left\{-\sin\lfloor\alpha\,\pi\delta+\sin\frac{1}{-\sin\lfloor\alpha\,\pi\delta}\right.$$

$$\left.+\sin\left|\frac{1}{-\sin\lfloor\alpha\,\pi\delta}\right|\right\}+\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}\left\{\sin\lfloor n\,\pi\delta+\sin\frac{1}{\sin\lfloor n\,\pi\delta}\right.$$

$$\left.+\sin\left|\frac{1}{\sin\lfloor n\,\pi\delta}\right|\right\}$$

$$=\frac{1}{\lfloor\alpha}\left\{-\sin\lfloor n\,\pi\delta\right\}+\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}\left\{\sin\lfloor n\,\pi\delta\right.$$

$$\left.+2\sin\frac{1}{\sin\lfloor n\,\pi\delta}\right\}$$

which oscillates as δ approaches zero.

On the left we have;

$$f\left(\tfrac{n}{q}-\delta\right)-f\left(\tfrac{n}{q}\right)=\frac{1}{\lfloor\alpha}\left\{\sin\lfloor\alpha\,\pi\delta+\sin\frac{1}{\sin\lfloor n\,\pi\delta}\right.$$

$$\left.+\sin\left|\frac{1}{\sin\lfloor n\,\pi\delta}\right|\right\}+\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}\left\{-\sin\lfloor n\,\pi\delta+\sin\frac{1}{-\sin\lfloor n\,\pi\delta}\right.$$

$$\left.+\sin\left|\frac{1}{-\sin\lfloor n\,\pi\delta}\right|\right\}$$

$$=\frac{1}{\lfloor\alpha}\left\{\sin\lfloor\alpha\,\pi\delta+2\sin\frac{1}{\sin\lfloor n\,\pi\delta}\right.$$

$$\left.+\sum_{n=\alpha+1}^{\infty}\frac{1}{\lfloor n}\left\{-\sin\lfloor n\,\pi\delta\right\}\right.$$

which also oscillates as δ approaches

zero. Therefore $f(x)$, in these points has a discontinuity of the second kind upon both sides.

Summing up our results then, we may say;

$f(x)$ is continuous for all irrational values of x. At all rational points for which

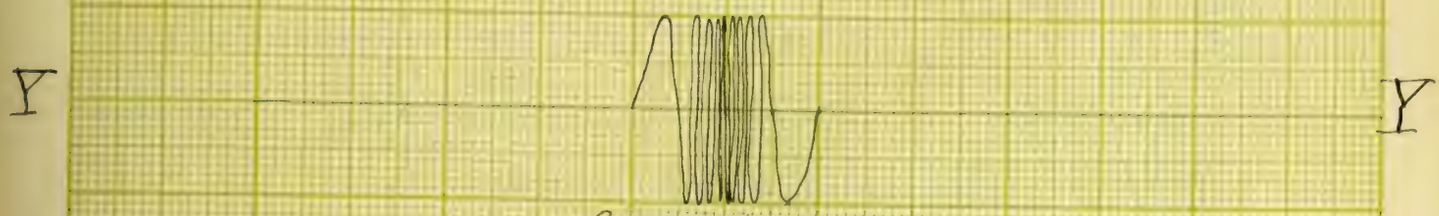
$$h\alpha = 2k'\pi$$

$f(x)$ has a discontinuity of the second kind upon the right and is continuous upon the left. At all rational points for which k is odd, i. e.

$$h\alpha = (2k'+1)\pi$$

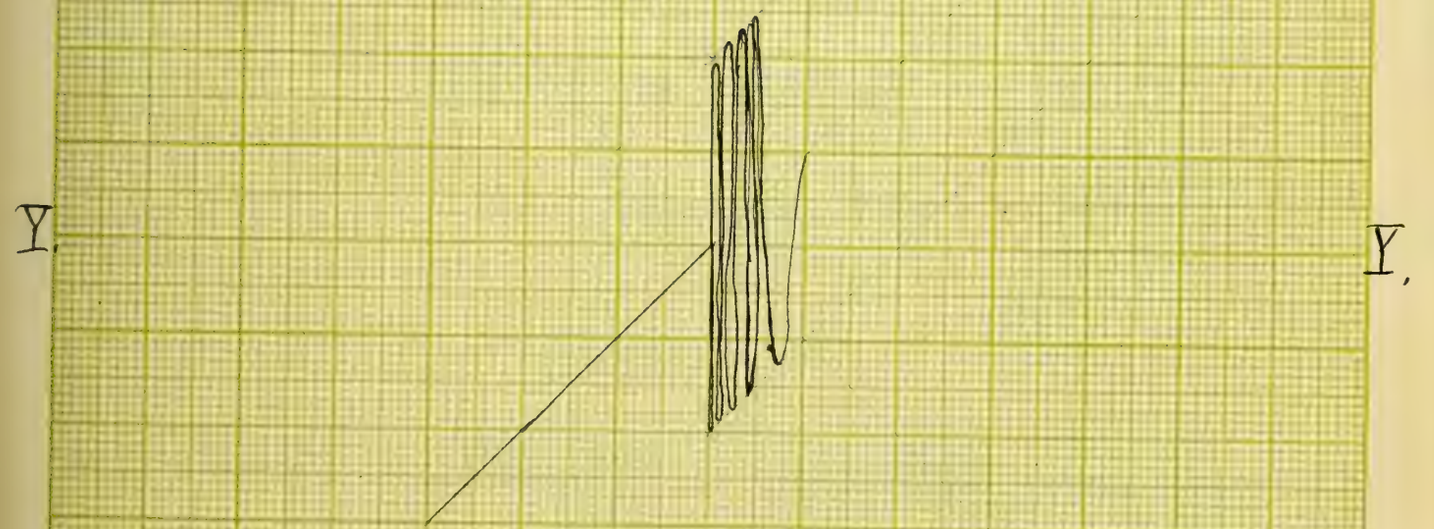
$f(x)$ has a discontinuity of the second kind upon both sides of the point in question.

$\varphi(y)$



Sketch showing
the behavior of the
function
$\varphi(y) = \sin 1/y$
near the point $y = 0$.

$\varphi(y)$.



Sketch showing
the behavior of the function

$\varphi(y) = y + \sin 1/y + \sin |1/y|$

in the neighborhood of the
point
$y = 0$.

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(5). Given; (See page 139).

$$\varphi(y) \equiv \frac{1}{1+e^{1/y}} + 2\sin y + \sin|1/y|$$

where we define $\varphi(y) = 0$ for $y = 0$. We have then for positive y,

$$\varphi(y) \equiv \frac{1}{1+e^{1/y}} + 2\sin 1/y$$

The first term on the right of this equation is always a finite and continuous function as y approaches zero, but the second term oscillates between -2 and $+2$, hence $\varphi(y)$ oscillates as y approaches zero and has a discontinuity of the second kind upon the right of the point $y = 0$. For negative y we have,

$$\varphi(y) = \frac{1}{1+e^{1/y}} + 2\sin\frac{1}{-y} + \sin|\tfrac{1}{-y}|$$

$$= \frac{1}{1+\frac{1}{e^{1/y}}} - \sin 1/y + \sin 1/y$$

$$= \frac{1}{1+\frac{1}{e^{1/y}}}$$

which is finite and continuous for $y < 0$ but approaches 1 as a limit as y approaches zero. Therefore since $\varphi(y) = 0$ for $y = 0$, $\varphi(y)$ has a discontinuity of the first kind upon the left of the point $y = 0$.

Summing $f(x)$ we get, for

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\frac{1}{\sin^m n\pi x}}} + \sin \frac{1}{\sin^m n\pi x} + \sin \left| \frac{1}{\sin^m n\pi x} \right| \right\}$$

Without further proof we may say that $f(x)$ is continuous at every irrational point, since $\varphi(y)$ is continuous for $y \neq 0$. In order to examine the rational points we have the test.

$$f\left(\tfrac{h}{g} \pm \delta\right) - f\left(\tfrac{h}{g}\right) = \sum_{n=1}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\frac{1}{\sin^m n\pi\left(\frac{h}{g}\pm\delta\right)}}} - \frac{1}{1+e^{\frac{1}{\sin^m n\pi \frac{h}{g}}}} \right.$$

$$+ \sin \frac{1}{\sin^m n\pi\left(\tfrac{h}{g}\pm\delta\right)} - \sin \frac{1}{\sin^m n\pi \tfrac{h}{g}} + \sin \left| \frac{1}{\sin^m n\pi\left(\tfrac{h}{g}\pm\delta\right)} \right|$$

$$\left. - \sin \left| \frac{1}{\sin^m n\pi \tfrac{h}{g}} \right| \right\}.$$

Since $\varphi(y)$ is continuous for $y \neq 0$, all the terms will disappear up to the term where

$$n = \alpha$$

as before in example (4) page 127.

We have then, reducing and arranging the terms

$$f\left(\tfrac{h}{g} \pm \delta\right) - f\left(\tfrac{h}{g}\right) = \sum_{n=\alpha}^{\infty} \frac{1}{n^m} \left\{ \frac{1}{1+e^{\pm \frac{1}{\sin^m n\pi\delta}}} + \sin \frac{1}{\pm \sin^m n\pi\delta} \right.$$

$$+ \sin \left| \frac{1}{\pm \sin^m n\pi\delta} \right| - \left(\frac{1}{1+e^{\frac{1}{\sin^m n\pi \frac{h}{g}}}} + \sin \frac{1}{\sin^m n\pi \tfrac{h}{g}} \right.$$

$$\left. \left. + \sin \left| \frac{1}{\sin^m n\pi \tfrac{h}{g}} \right| \right) \right\}$$

But the expression after the $-$ sign in

this fraction there is simply $\psi(0)$ in every term since $\sin m\pi \frac{k}{q} - \sin k\pi = 0$ in every term. Our expression becomes, then, dropping these terms,

$$f(\tfrac{k}{q} \pm \delta) - f(\tfrac{k}{q}) = \sum_{n=\alpha}^{\infty} \frac{1}{m} \left\{ \frac{1}{1 + e^{\pm \frac{1}{\sin m\pi\delta}}} + \sin \pm \frac{1}{\sin m\pi\delta} \right.$$

$$\left. + \sin \left| \pm \frac{1}{\sin m\pi\delta} \right| \right\}.$$

Now let us examine this for $-\delta$ and $+\delta$ and when

$$k \alpha = 2k'q$$

and when

$$k \alpha = (2k'+1)q.$$

First, let k be even; i.e.

$$k\alpha = 2k'q.$$

We have then

$$f(\tfrac{k}{q} + \delta) - f(\tfrac{k}{q}) = \sum_{n=\alpha}^{\infty} \frac{1}{m} \left\{ \frac{1}{1 + e^{\frac{1}{\sin m\pi\delta}}} + \sin \frac{1}{\sin m\pi\delta} \right.$$

$$\left. + \sin \left| \frac{1}{\sin m\pi\delta} \right| \right\}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{m} \left\{ \frac{1}{1 + e^{\frac{1}{\sin m\pi\delta}}} + 2 \cdot \sin \frac{1}{\sin m\pi\delta} \right\}$$

Since $\sin m\pi\delta$ approaches zero as δ approaches zero this expression oscillates and $f(x)$ has a discontinuity of the second kind upon the right of the point in

question. For $-\delta$ we have,

$$f(\tfrac{\pi}{4}-\delta) - f(\tfrac{\pi}{4}) = \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \left\{ \frac{1}{1+e^{-\sin n\pi\delta}} + \sin \frac{1}{-\sin \lfloor m \pi\delta} \right.$$

$$\left. + \sin \left| \frac{1}{-\sin \lfloor m \pi\delta} \right| \right\}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \left\{ \frac{1}{1+e^{-\sin \lfloor m \pi\delta}} - \sin \frac{1}{+\sin \lfloor m \pi\delta} \right.$$

$$\left. + \sin \frac{1}{+\sin \lfloor m \pi\delta} \right\}$$

$$= \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n} \left\{ \frac{1}{1+e^{-\sin \lfloor m \pi\delta}} \right\}$$

Since $\sin \lfloor m \pi\delta$ approaches zero as δ approaches zero the limit of this expression is

$$\sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$$

for the limit of

$$\frac{1}{1+e^{\frac{1}{y}}}$$

is 1 as y approaches zero. Therefore $f(x)$ has a discontinuity of the first kind upon the left of the point in question.

Now let κ be odd; i. e.

$$\text{If } \lfloor \alpha = (2k'+1)\,\pi$$

We have then for $+\delta$,

$$f\left(\frac{k}{q}+\delta\right)-f\left(\frac{k}{q}\right)=\frac{1}{\Gamma\alpha}\left\{\frac{1}{1+e^{\frac{1}{\sin\pi\beta}\pi\delta}}+\sin\frac{1}{\sin\pi\beta}\pi\delta+\sin\left|\frac{1}{\sin\pi\beta}\pi\delta\right|\right\}$$

$$+\sum_{n=\alpha+1}^{\infty}\frac{1}{\Gamma n}\left\{\frac{1}{1+e^{\frac{1}{\sin\pi\beta}\pi\delta}}+\sin\frac{1}{\sin\pi\beta}\pi\delta+\sin\left|\frac{1}{\sin\pi\beta}\pi\delta\right|\right\}$$

$$=\frac{1}{\Gamma\alpha}\left\{\frac{1}{1+e^{-\frac{1}{\sin\pi\beta}\pi\delta}}-\sin\frac{1}{\sin\pi\beta}\pi\delta+\sin\frac{1}{\sin\pi\beta}\pi\delta\right\}$$

$$+\sum_{n=\alpha+1}^{\infty}\frac{1}{\Gamma n}\left\{\frac{1}{1+e^{\frac{1}{\sin\pi\beta}\pi\delta}}+\sin\frac{1}{\sin\pi\beta}\pi\delta+\sin\left|\frac{1}{\sin\pi\beta}\pi\delta\right|\right\}$$

$$=\frac{1}{\Gamma\alpha}\left\{\frac{1}{1+e^{-\frac{1}{\sin\pi\beta}\pi\delta}}\right\}+\sum_{n=\alpha+1}^{\infty}\frac{1}{\Gamma n}\left\{\frac{1}{1+e^{+\frac{1}{\sin\pi\beta}\pi\delta}}\right.$$

$$\left.+2\sin\frac{1}{\sin\pi\beta}\pi\delta\right\}$$

Since the last term in this expression oscillates, the whole must oscillate and $f(x)$ has a discontinuity of the second kind on the right of this point. On the other side of the point we have,

$$f\left(\frac{k}{q}-\delta\right)-f\left(\frac{k}{q}\right)=\frac{1}{\Gamma\alpha}\left\{\frac{1}{1+e^{\frac{1}{\sin\pi\beta}\pi\delta}}+\sin\frac{1}{\sin\pi\beta}\pi\delta\right.$$

$$\left.+\sin\left|\frac{1}{\sin\pi\beta}\pi\delta\right|\right\}+\sum_{n=\alpha+1}^{\infty}\frac{1}{\Gamma n}\left\{\frac{1}{1+e^{-\frac{1}{\sin\pi\beta}\pi\delta}}\right.$$

$$\left.+\sin-\frac{1}{\sin\pi\beta}\pi\delta+\sin\left|-\frac{1}{\sin\pi\beta}\pi\delta\right|\right\}$$

$$=\frac{1}{\Gamma\alpha}\left\{\frac{1}{1+e^{\frac{1}{\sin\pi\beta}\pi\delta}}+2\sin\frac{1}{\sin\pi\beta}\pi\delta\right\}$$

$$+\sum_{n=\alpha+1}^{\infty}\frac{1}{\Gamma n}\left\{\frac{1}{1+e^{-\frac{1}{\sin\pi\beta}\pi\delta}}\right\}$$

We have the term $2/\alpha \cdot \sin \frac{1}{2}n\pi \cdot \sin n\pi\delta$ in this expression also. Therefore $f(x)$ has a discontinuity of the second kind upon the left of the point as well as upon the right.

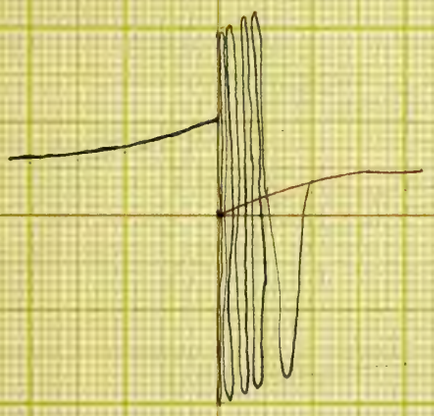
We have only taken up the case in which α is an even number but it may be seen by reference to page 16 and those following that the results obtained when α is odd are exactly the same as when it is even. Hence we have proven that $f(x)$ has, in the rational points for which

$$p/\alpha = 2k'/q,$$

a discontinuity of the first kind upon the left and of the second kind upon the right. For those points for which

$$p/\alpha = (2k'+1)/q$$

$f(x)$ has a discontinuity of the second kind upon both sides, and $f(x)$ is continuous for all irrational values of x.



Sketch
Showing the behavior of
the function

$$\varphi(y) \equiv \frac{1}{1+e^{1/y}} + \sin 1/y + \sin|1/y|$$

in the neighborhood
of the point
$y = 0$.

Section IX

This section will be devoted to problems illustrating the method set forth in section II. In order that we may more readily compare the results of the two methods some of the same problems are discussed here which were discussed in the last section and at the end of section I.

(1) Given;

$$\varphi(y) = \lim_{m=\infty} (y)^{2/m}$$

Then we have

$$\varphi(y) = 1 \quad \text{for} \quad y \neq 0$$
$$\varphi(y) = 0 \quad " \quad y = 0.$$

letting $s = 2$, and forming $f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} \lim_{m=\infty} \frac{(\sin^2 n\pi x)^{\frac{1}{m}}}{n^2}$$

Without further proof we may say that since $\varphi(y)$ is continuous for y different from zero, $f(x)$ is continuous for all irrational values of x. For the rational values we must apply the test for continuity, namely,-

$$f\left(\tfrac{k}{q}\pm\delta\right)-f\left(\tfrac{k}{q}\right)=\sum_{n=1}^{\infty}\frac{\lim_{m=\infty}\left[\left\{2\sin^2 n\pi\left(\tfrac{k}{q}\pm\delta\right)\right\}^{\frac{1}{m}}-\left\{2\sin^2 n\pi\,\tfrac{k}{q}\right\}^{\frac{1}{m}}\right]}{n^2}$$

This summation may now be divided into two parts, as follows,

$$f\left(\tfrac{k}{q}\pm\delta\right)-f\left(\tfrac{k}{q}\right)=\sum_{n=1}^{\infty}\frac{\lim_{m=\infty}\left[\left\{2\sin^2 n\pi\left(\tfrac{k}{q}\pm\delta\right)\right\}^{\frac{1}{m}}-\left\{2\sin^2 n\pi\,\tfrac{k}{q}\right\}^{\frac{1}{m}}\right]}{n^2}\quad;\ fn\neq kq$$

$$+\sum_{n=1}^{\infty}\frac{\lim_{m=\infty}\left[\left\{\sin^2 n\pi\left(\tfrac{k}{q}\pm\delta\right)\right\}^{\frac{1}{m}}-\left\{\sin^2 n\pi\,\tfrac{k}{q}\right\}^{\frac{1}{m}}\right]}{n^2}$$

$fn=kq$

The first summation represents those terms in which fn is not an integral multiple of q and the second summation contains those terms in which fn is such a multiple. The first summation will become zero in the limit and we may drop it from our equation. We may write then

$$f\left(\tfrac{k}{q}\pm\delta\right)-f\left(\tfrac{k}{q}\right)=\sum_{n=1}^{\infty}\frac{\lim_{m=\infty}\left[\left\{\sin^2 n\pi\left(\tfrac{k}{q}\pm\delta\right)\right\}^{\frac{1}{m}}-\left\{\sin^2 n\pi\,\tfrac{k}{q}\right\}^{\frac{1}{m}}\right]}{n^2}\quad;\ fn=kq$$

$$=\sum_{n=1}^{\infty}\frac{\lim_{m=\infty}\left[\left\{\sin^2 n\pi\left(\tfrac{k}{q}\pm\delta\right)\right\}^{\frac{1}{m}}\right]}{n^2}\quad;\ fn=kq$$

since $\sin n\pi\,\frac{k}{q}$ is equal to zero in every term. This may be further sim-

simplified now, and we have

$$f\left(\frac{p}{q} \pm \delta\right) - f\left(\frac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{\lim_{m=\infty} \left[\left\{ (\pm 2 i \sin n\pi\delta)^2 \right\}^{\frac{1}{m}} \right]}{n^2}$$

if $n = kq$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{if } n \neq kq$$

Therefore $f(x)$ has a discontinuity of the second kind upon both sides of the point

$$x = \frac{p}{q}$$

The fact that p may be even or odd will make no difference here since we have the square of $\sin n\pi\delta$, and this is positive no matter whether $\sin n\pi\delta$ itself is positive or negative. (For plot of this function see page 41 as it is identical with that of the function whose graph is given there).

(2). Given;

$$\varphi(y) = y + \sin \tfrac{1}{y} + \sin \left|\tfrac{1}{y}\right|$$

For a discussion of this function see section VIII pages 126 and 131.

Forming $f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ \sin n\pi x + \sin \frac{1}{\sin n\pi x} + \sin \left| \frac{1}{\sin n\pi x} \right| \right\}$$

This function is continuous for all irrational points since $\varphi(y)$ is continuous for y different from zero.

For the rational points we must apply the test as before,

$$f\left(\tfrac{p}{q} \pm \delta\right) - f\left(\tfrac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ \sin n\pi\left(\tfrac{p}{q} \pm \delta\right) + \sin \frac{1}{\sin n\pi\left(\tfrac{p}{q} \pm \delta\right)} \right.$$

$$+ \sin \left| \frac{1}{\sin n\pi \tfrac{p}{q}} \right| - \left(\sin n\pi \tfrac{p}{q} \right.$$

$$\left. \left. + \sin \frac{1}{\sin n\pi \tfrac{p}{q}} + \sin \left| \frac{1}{\sin n\pi \tfrac{p}{q}} \right| \right) \right\}$$

All the terms in this summation in which

$$pn \pm kq$$

will drop out and we shall have

$$f\left(\tfrac{p}{q} \pm \delta\right) - f\left(\tfrac{p}{q}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ \pm \sin n\pi\delta + \sin \frac{1}{\pm \sin n\pi\delta} \right.$$

$$+\sin\left|\frac{1}{\pm\sin n\pi\delta}\right| - \left(\sin k\pi + \sin\frac{1}{\sin k\pi}\right.$$

$$\left. + \sin\frac{1}{\sin k\pi}\right)\Big\} \qquad ;\text{if } n = kq$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\pm\sin n\pi\delta + \sin\frac{1}{\pm\sin n\pi\delta}\right.$$

$$\left. + \sin\left|\frac{1}{\pm\sin n\pi\delta}\right|\right\} \qquad \text{if } n = kq.$$

Now let us examine the point in which p is an even number. We have seen (page 34) that in this case k is always even and we have

$$\sin n\pi\left(\tfrac{p}{q}+\delta\right) = \sin(2k'\pi + n\pi\delta)$$
$$= \sin n\pi\delta$$

and

$$\sin n\pi\left(\tfrac{p}{q}-\delta\right) = \sin(2k'\pi - n\pi\delta)$$
$$= -\sin n\pi\delta$$

Then our equation above becomes for $+\delta$, i.e. on the right of $x = \frac{p}{q}$

$$f\left(\tfrac{p}{q}+\delta\right) - f\left(\tfrac{p}{q}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\sin n\pi\delta + \sin\frac{1}{\sin n\pi\delta}\right.$$

$$\left. + \sin\left|\frac{1}{\sin n\pi\delta}\right|\right\} \qquad ;\text{if } n = kq.$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\sin n\pi\delta + 2\sin\frac{1}{\sin n\pi\delta}\right\}$$

$$;\text{if } n = kq$$

Since this contains a term in $\sin\frac{1}{\sin n\pi\delta}$ it will oscillate as δ approaches zero and $f(x)$ has a discontinuity of the second kind upon the right of this point. On the left we have

$$f\left(\tfrac{p}{q}-\delta\right)-f\left(\tfrac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{-\sin n\pi\delta+\sin\frac{1}{\sin n\pi\delta}\right.$$

$$\left.+\sin\left|\frac{1}{\sin n\pi\delta}\right|\right\} \qquad ;\text{if } n=kq$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{-\sin n\pi\delta-\sin\frac{1}{\sin n\pi\delta}\right.$$

$$\left.+\sin\frac{1}{\sin n\pi\delta}\right\} \qquad ;\text{if } n=kq$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{-\sin n\pi\delta\right\} \qquad ;\text{if } n=kq$$

This approaches zero as δ approaches zero and hence $f(x)$ is continuous on the left of the point in question.

When p is odd we found that k is sometimes odd and sometimes even. When k is odd we have

$$\sin n\pi\left(\tfrac{p}{q}+\delta\right)=\sin\{(2k+1)\pi+n\pi\delta\}$$
$$=-\sin n\pi\delta$$

and

$$\sin n\pi\left(\tfrac{p}{q}-\delta\right)=\sin\{(2k+1)\pi-n\pi\delta\}$$
$$=\sin n\pi\delta$$

We have then for our summation, a priori

$$f(\tfrac{n}{q}+\delta) - f(\tfrac{n}{q}) = \sum_{n=1}^{\infty} \frac{1}{n^2}\Big\{ \pm \sin n\pi\delta + \sin \pm \sin n\pi\delta + \sin \Big| \pm \sin n\pi\delta \Big| \Big\} \quad ; n = kq$$

Since it is sometimes odd and sometimes even we have

$$f(\tfrac{n}{q}+\delta) - f(\tfrac{n}{q}) = \sum_{n=1}^{\infty} \frac{1}{n^2}\Big\{ \sin n\pi\delta + \sin \frac{1}{\sin n\pi\delta} + \sin \Big| \frac{1}{\sin n\pi\delta} \Big| \Big\} + \sum_{n=1}^{\infty} \frac{1}{n^2}\Big\{ -\sin n\pi\delta + \sin \frac{1}{-\sin n\pi\delta} + \sin \Big| \frac{1}{\sin n\pi\delta} \Big| \Big\}$$

The first summation contains the terms in which k is even and the second those in which k is odd. These will reduce now in the following manner;

$$f(\tfrac{n}{q}+\delta) - f(\tfrac{n}{q}) = \sum_{n=1}^{\infty} \frac{1}{n^2}\Big\{ \sin n\pi\delta + 2\sin \frac{1}{2\sin n\pi\delta} \Big\} \quad ; n = (2k')q$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n^2}\Big\{ -\sin n\pi\delta \Big\} \quad ; n = (2k'+1)q.$$

The first summation contains the term $2\sin \frac{1}{\sin n\pi\delta}$, hence $f(x)$ has a discontinuity of the second kind upon the right of this point.

On the left we get,

$$f\left(\frac{h}{k}-\delta\right)-f\left(\frac{h}{k}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}\{-\sin n\pi\delta\}\quad ;\text{for } n=2k'k$$

$$+\sum_{n=1}^{\infty}\frac{1}{n^2}\{\sin n\pi\delta+2\sin\tfrac{1}{2}n\pi\delta\}$$

$$\text{for } n=(2k'+1)k$$

Here the second summation contains the term $2\sin\frac{1}{2}n\pi\delta$ and $f(x)$ has a discontinuity and the second kind upon the left of the point $\frac{h}{k}$ as specified.

We have seen then that $f(x)$ is continuous at all irrational points, at the rational points in which k is even $f(x)$ has a discontinuity of the second kind upon the right and is continuous upon the left, and at the points where k is odd has a discontinuity of the second kind upon both sides of the point.

Section X.

The problems given in this section are to illustrate Cantor's method of condensation, discussed in full in section III.

(1) Given;

$$\varphi(y) = \frac{1}{1+e^{1/y}} \qquad \text{(See page 123).}$$

This function is continuous and finite for all values of y different from zero, but at the point

$$y = 0$$

it has a discontinuity of the first kind upon the left and is continuous upon the right, when we define $\varphi(0) = 0$.

letting

$$c_n = \frac{1}{n!}$$

and taking the rational numbers as our menge of w's we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{1}{1+e^{\frac{1}{x-w_n}}} \right) \quad ; w_n = \frac{n}{8}$$

Since w_n is always a rational number, $(x - w_n)$ can not be zero for any irrational number and hence $f(x)$ is continuous for all irrational values of x.

At the rational points we have

$$f(w_k \pm \delta) - f(w_k) = \sum_{n=1}^{\infty} \frac{1}{l_n} \left\{ \frac{1}{1+e^{w_k - w_n \pm \delta}} - \frac{1}{1+e^{w_k - w_n}} \right\}$$

$$= \sum_{n=1}^{k-1} \frac{1}{l_n} \left\{ \frac{1}{1+e^{w_k - w_n \pm \delta}} - \frac{1}{1+e^{w_k - w_n}} \right\}$$

$$+ \frac{1}{l_k} \left\{ \frac{1}{1+e^{w_k - w_k \pm \delta}} - \frac{1}{1+e^{w_k - w_n}} \right\}$$

$$+ \sum_{n=k+1}^{\infty} \frac{1}{l_n} \left\{ \frac{1}{1+e^{w_k - w_n \pm \delta}} - \frac{1}{1+e^{w_k - w_n}} \right\}$$

Since

$$w_k - w_n \neq 0$$

in every term of the two summations those terms will all disappear in the limit and we have

$$f(w_k \pm \delta) - f(w_k) = \frac{1}{l_k} \left\{ \frac{1}{1+e^{w_k - w_k \pm \delta}} - \frac{1}{1+e^{w_k - w_k}} \right\}$$

$$= \frac{1}{l_k} \left\{ \frac{1}{1+e^{\pm \delta}} \right\}$$

Then for $+\delta$ we have

$$f(w_k + \delta) - f(w_k) = \frac{1}{l_k} \left\{ \frac{1}{1+e^{\delta}} \right\}$$

which can be made less than some arbitrarily chosen small positive quantity τ. Therefore $f(x)$ is continuous on the right of the point.

For $-\delta$ we have;

$$f(1/k-\delta)-f(1/k)=\frac{1}{1+e^{1/\delta}}.$$

The limit of this expression as δ approaches zero is $\frac{1}{1+e}$, hence $f(x)$ has a discontinuity of the first kind upon the left of the point. We may say then that in each of the points

$$x=1/k$$

$f(x)$ has the same kind of discontinuity that $y(y)$ has in the point

$$y=0,$$

namely, a discontinuity of the first kind on the left and continuous right.

(2) Given; $\varphi(y) = \dfrac{e^{\frac{1}{y}}-1}{e^{\frac{1}{y}}+1}$ (See page 153)

By examining this function as y approaches zero we find that $\varphi(y)$ approaches 1 as y approaches zero through positive values, and $\varphi(y)$ approaches -1 as y approaches zero through negative values. If now we define $\varphi(y) = 0$ for $y = 0$, as we are at liberty to do, $\varphi(y)$ will have a discontinuity of the first kind upon both sides of the point $y = 0$. Forming $f(x)$ now we have:

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{|n|} \left\{ \frac{e^{\frac{1}{x-w_n}} - 1}{e^{\frac{1}{x-w_n}} + 1} \right\}$$

where w_n represents a sequence of irrational numbers. Since $\varphi(y)$ is continuous for $y \neq 0$, $f(x)$ is continuous at all irrational points. At the rational points we have,

$$f(w_k \pm \delta) - f(w_k) \equiv \sum_{n=1}^{k-1} \frac{1}{|n|} \left\{ \frac{e^{\frac{1}{w_k \pm \delta - w_n}} - 1}{e^{\frac{1}{w_k \pm \delta - w_n}} + 1} - \frac{e^{\frac{1}{w_k - w_n}} - 1}{e^{\frac{1}{w_k - w_n}} + 1} \right\}$$

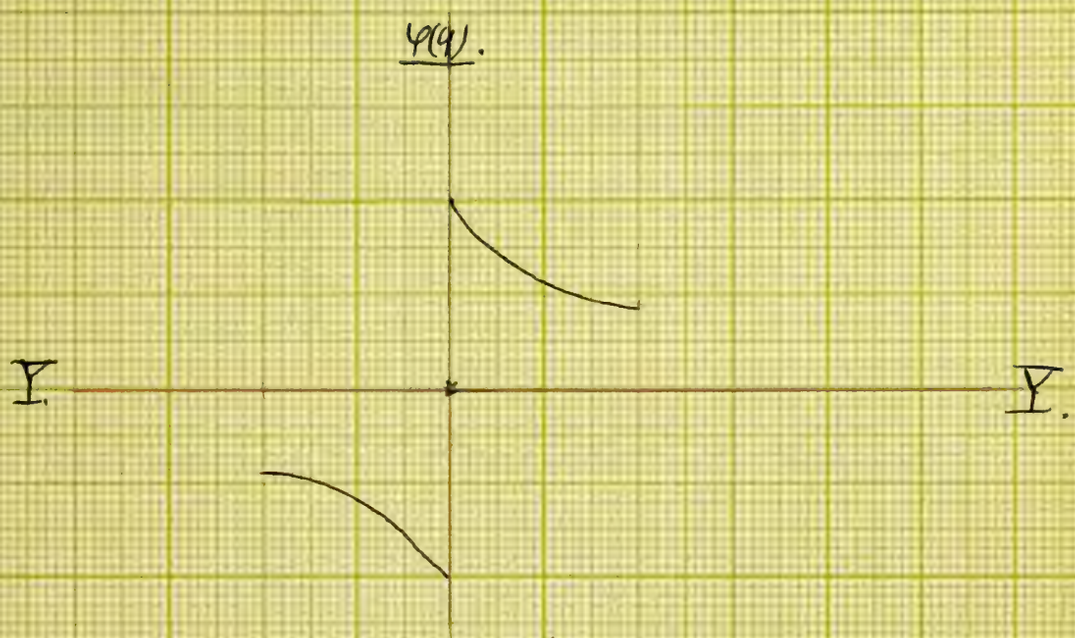
$$+ \frac{1}{|k|} \left\{ \frac{e^{\pm\frac{1}{\delta}} - 1}{e^{\pm\frac{1}{\delta}} + 1} \right\} + \sum_{n=k+1}^{\infty} \frac{1}{|n|} \left\{ \frac{e^{\frac{1}{w_k - w_n \pm \delta}} - 1}{e^{\frac{1}{w_k - w_n \pm \delta}} + 1} - \frac{e^{\frac{1}{w_k - w_n}} - 1}{e^{\frac{1}{w_k - w_n}} + 1} \right\}$$

Since $(x - w_n) \neq 0$ in every term of the two summations, they will each become zero

in the limit and we shall have

$$f(1/k \pm \delta) - f(1/k) = \frac{1}{1k}\left\{\frac{e^{\pm\delta}-1}{e^{\pm\delta}+1}\right\}.$$

The limit of this expression is different for $+\delta$ and $-\delta$, and neither limit is equal to zero. Therefore $f(x)$ has a discontinuity of the first kind upon both sides of the point in question i.e. at every rational point.



Graph of
the function

$$\varphi(y) \equiv \frac{e^{y/4} - 1}{e^{y/4} + 1}.$$

Scale - 2 Squares = 1 Unit.

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Section XI.

In this section, the problems are to illustrate the principles set forth in section V.

(1). Given;

$$\varphi(y) \equiv \frac{e^{\frac{1}{y}}}{1+e^{\frac{1}{y}}}$$ (See page 158)

The limit of this expression as y approaches zero from the positive side is

$$\frac{e^{\frac{1}{y}}}{1+e^{\frac{1}{y}}} = \frac{1}{\frac{1}{e^{\frac{1}{y}}}+1} = 1$$

From the negative side we have

$$\frac{e^{-\frac{1}{y}}}{e^{-\frac{1}{y}}+1} = \frac{\frac{1}{e^{\frac{1}{y}}}}{1+\frac{1}{e^{\frac{1}{y}}}} = 0.$$

Therefore $\varphi(y)$ has a discontinuity of the first kind upon the right of the point $y=0$, if we define $\varphi(y)=0$, when $y=0$. Forming $f(x)$ now we have,

$$f(x) = \frac{e^{\frac{1}{\sin m\pi x}}}{1+e^{\frac{1}{\sin m\pi x}}}$$

$f(x)$, then, is continuous for every

irrational value of x for the rational values of x for which

$$k = \kappa x$$

we have

$$f\left(\frac{h}{\kappa}+\delta\right) - f\left(\frac{h}{\kappa}\right) \equiv \left\{ \frac{e^{\frac{1}{\sin m\pi\left(\frac{h}{\kappa}+\delta\right)}}}{1+e^{\frac{1}{\sin m\pi\left(\frac{h}{\kappa}+\delta\right)}}} - \frac{e^{\frac{1}{\sin m\pi\frac{h}{\kappa}}}}{e^{\frac{1}{\sin m\pi\frac{h}{\kappa}}}+1} \right\}$$

If κ is odd we have

$$f\left(\frac{h}{\kappa}+\delta\right) - f\left(\frac{h}{\kappa}\right) \equiv \left\{ \frac{e^{-\frac{1}{\sin m\pi\delta}}}{1+e^{-\frac{1}{\sin m\pi\delta}}} \right\}$$

and the limit of this expression, as δ approaches zero, is zero. On the other side of this point we have

$$f\left(\frac{h}{\kappa}-\delta\right) - f\left(\frac{h}{\kappa}\right) \equiv \left\{ \frac{e^{\frac{1}{\sin m\pi\delta}}}{1+e^{\frac{1}{\sin m\pi\delta}}} \right\}$$

and the limit of this expression, as δ approaches zero, is 1. Therefore $f(x)$ is continuous on the right and has a discontinuity of the first kind when we left of the point in question.

If κ is even we have,

$$f\left(\frac{h}{\kappa}+\delta\right) - f\left(\frac{h}{\kappa}\right) \equiv \frac{e^{\frac{1}{\sin m\pi\delta}}}{1+e^{\frac{1}{\sin m\pi\delta}}}$$

and the limit of this expression is 1, as δ approaches zero. On the other side we have

$$f(\tfrac{1}{q}-\delta) - f(\tfrac{1}{q}) = \left\{ \frac{e^{-\frac{1}{\sin m\pi\delta}}}{1+e^{-\frac{1}{\sin m\pi\delta}}} \right\}$$

The limit of this expression, as δ approaches zero, is zero. Therefore $f(x)$ is continuous left and has a discontinuity of the first kind upon the right of this point.

(2) Given;

$$\varphi(y) = \lim_{m=\infty} \{1 - (\cos \pi y)^{2/m}\}$$ (See page 158)

Therefore $\varphi(y) = 0$ when $y = 0$, and $\varphi(y) = 1$ for every other value of y. We have then

$$f(x) = \lim_{m=\infty} \{\lim_{n=\infty} (1 - \overline{\cos(\sin n! \pi x)}^{2n})^{2/m}\}$$

Then $f(x) = 0$ for every rational x and $f(x) = 1$ for every irrational value. Hence $f(x)$ is discontinuous at every point. For plot of this function see page 158.

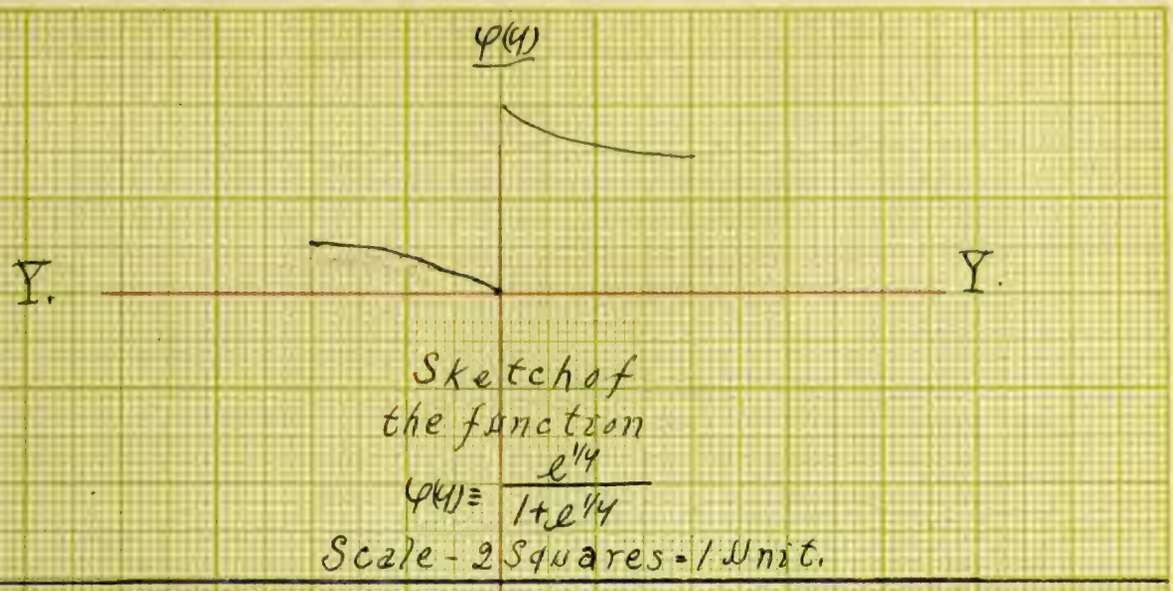
(3). Given;

$$\varphi(y) = \lim_{m=\infty} (y)^{2/m}$$ (See page 159)

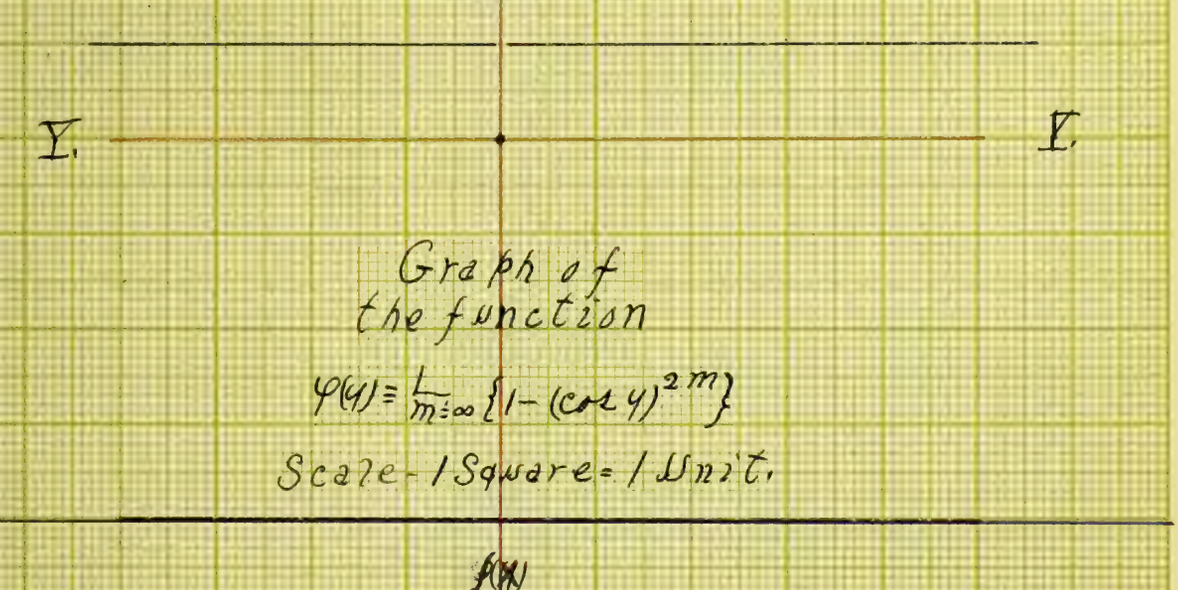
Then $\varphi(y) = 0$ when $y = 0$, and $\varphi(y) = 1$ for every other value of y. Forming $f(x)$ then, we have,

$$f(x) = \lim_{m=\infty} \left[\lim_{n=\infty} X\{\sin n! \pi x\}^{2/m}\right]$$

Therefore $f(x)$ is equal to zero for every rational value of x, and is equal to x for all irrational values. Hence $f(x)$ is discontinuous at every point except $x = 0$. For plot see page 159.



Sketch of
the function
$\varphi(y) \equiv \frac{e^{1/y}}{1+e^{1/y}}$
Scale - 2 Squares = 1 Unit.



Graph of
the function
$\varphi(y) \equiv \lim_{m=\infty} \{1-(\cos y)^{2m}\}$
Scale - 1 Square = 1 Unit.

Graph
of the function
$f(x) \equiv \lim_{m=\infty}\left[\lim_{n=\infty}\{1-\overline{\cos(\sin m\pi x)}^{2m}\}\right]$
Scale - 1 Square = 1 Unit.

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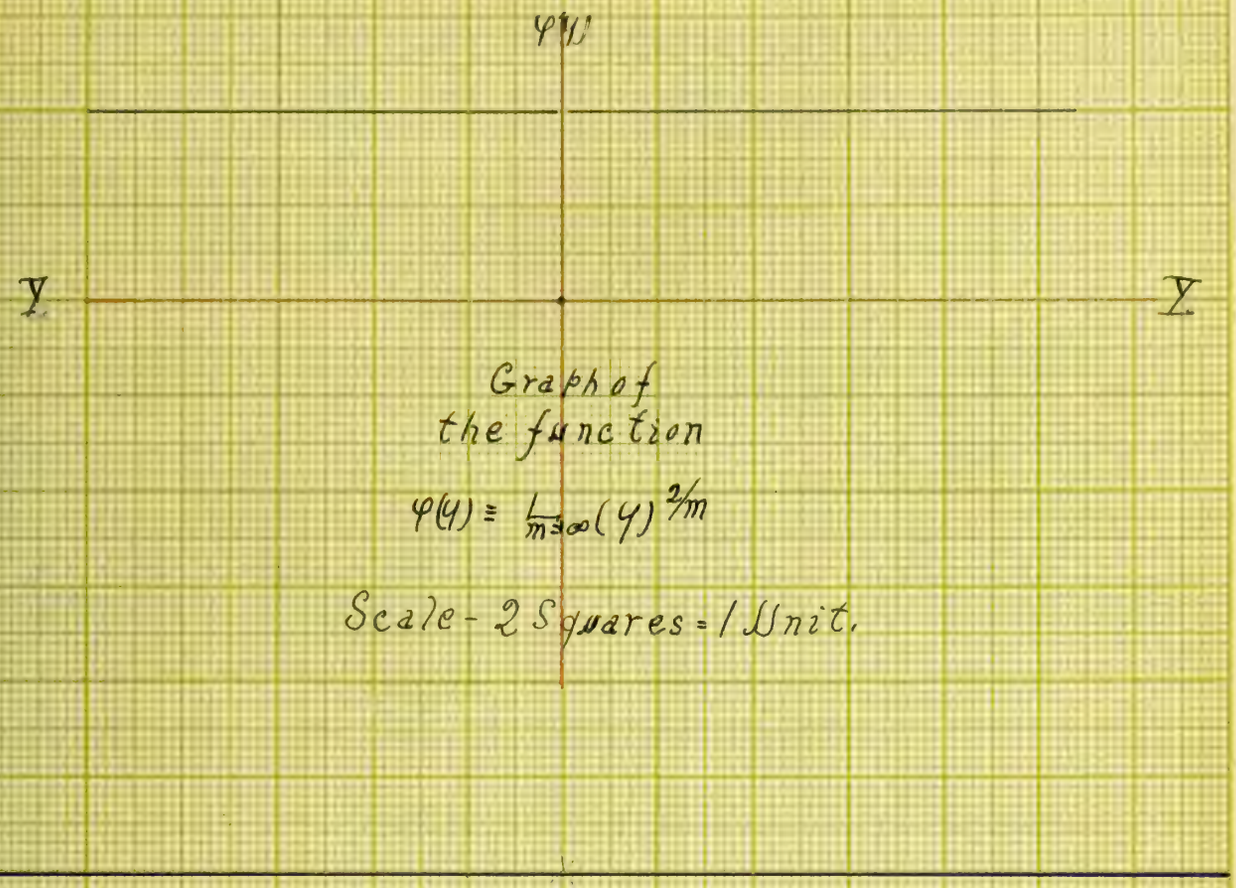
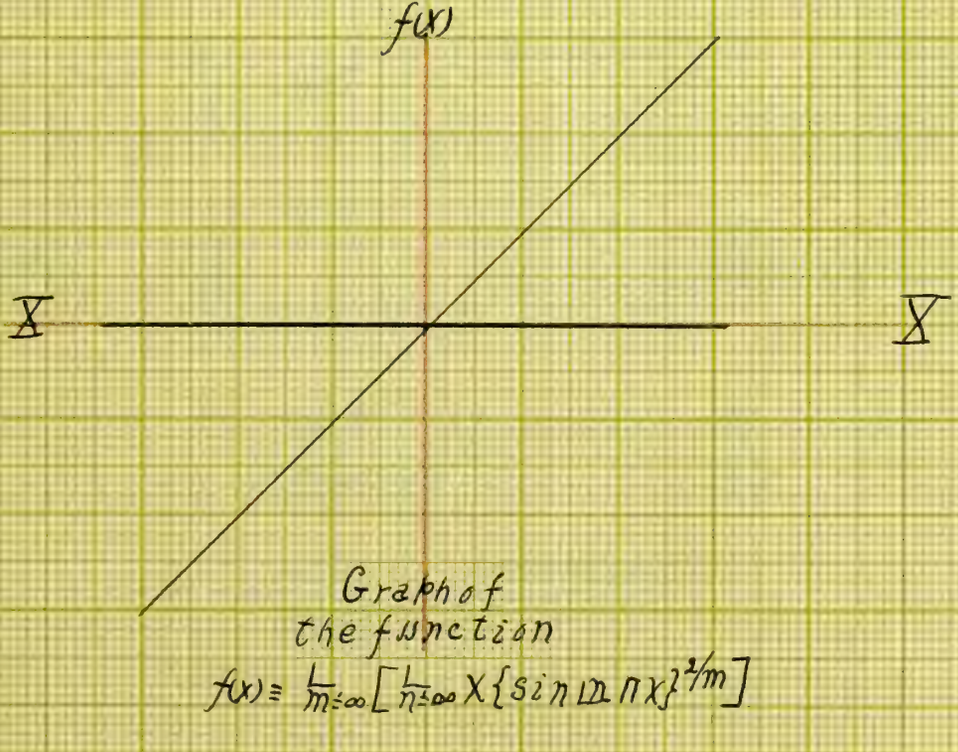


Graph of
the function

$$\varphi(y) \equiv \lim_{m=\infty} (y)^{2/m}$$

Scale - 2 Squares = 1 Unit.



Graph of
the function

$$f(x) \equiv \lim_{m=\infty} \left[\lim_{n=\infty} X \{ \sin \Pi . n x \}^{2/m} \right]$$

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Section XII

The problems in this section are to illustrate the theory discussed in section VI.

(1) Given;

$$\varphi(y) \equiv |y| \qquad \text{(See page 164).}$$

Examining this in regard to its derivative at the point $y=0$ we find that the derivative of $\varphi(y)$ has $+1$ as a limit upon the right and -1 as a limit upon the left. Therefore $\varphi(y)$ has no fixed derivative at the point $y=0$.

Forming $f(x)$ now we have;

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{(\lfloor n \rfloor)^2} |\sin \lfloor n \rfloor \pi x|$$

$$f'(x) \equiv \pi \sum_{n=1}^{\infty} \frac{1}{\lfloor n \rfloor} |\cos \lfloor n \rfloor \pi x|$$

for all irrational points. At the rational points we have

$$\frac{f(\frac{m}{q} \pm \delta) - f(\frac{m}{q})}{\pm \delta} = \sum_{n=1}^{\infty} \frac{1}{(\lfloor n \rfloor)^2} \left\{ \frac{|\sin \lfloor n \rfloor \pi (\frac{m}{q} \pm \delta)| - |\sin \frac{m}{q} \pi \lfloor n \rfloor|}{\pm \delta} \right\}$$

$$= \sum_{n=1}^{q-1} \frac{1}{(\lfloor n \rfloor)^2} \left\{ \frac{|\sin \lfloor n \rfloor \pi (\frac{m}{q} \pm \delta)| - |\sin \lfloor n \rfloor \pi \frac{m}{q}|}{\pm \delta} \right\}$$

$$+ \sum_{n=q}^{\infty} \frac{1}{(\lfloor n \rfloor)^2} \left\{ \frac{|\pm \sin \lfloor n \rfloor \delta \pi|}{\pm \delta} \right\}$$

Since the first summation extends to those terms for which $\sin \lfloor n \pi \frac{m}{q}$ is not zero we may write

$$\lim_{\delta \to 0} \frac{f(\frac{m}{q}+\delta) - f(\frac{m}{q})}{\delta} = \sum_{n=1}^{\alpha-1} \frac{1}{\lfloor n} \left| \cos \lfloor n \pi \frac{m}{q} \right|$$

$$+ \lim_{\delta \to 0} \sum_{n=\alpha}^{\infty} \frac{1}{(\lfloor n)^2} \frac{|\sin \lfloor n \pi \delta|}{\delta}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lfloor n} \left| \cos \lfloor n \pi \frac{m}{q} \right|$$

$$+ \pi \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$$

In like manner we may obtain

$$\lim_{\delta \to 0} \frac{f(\frac{m}{q}-\delta) - f(\frac{m}{q})}{-\delta} = \sum_{n=1}^{\alpha-1} \frac{1}{\lfloor n} \left| \cos \lfloor n \pi \frac{m}{q} \right|$$

$$- \sum_{n=\alpha}^{\infty} \frac{1}{\lfloor n}$$

Therefore, since the derivative of $f(x)$ is different upon the two sides of the point $x = \frac{m}{q}$, the derivative does not exist at that point, and we have a function which has no derivative at any rational point.

(2). Given:

$$\varphi(y) \equiv y \sin(\log y^2)$$

The derivative of $\varphi(y)$, namely

$$\varphi'(y) \equiv \sin(\log y^2) + y \cos(\log y^2) \cdot 2\frac{y}{y^2}$$

$$= \sin(\log y^2) + 2\cos(\log y^2)$$

$$= 2\sin\alpha + 2\cos\alpha \quad ; y = 0$$

is indeterminate, hence $\varphi'(y)$ does not exist at the point $y = 0$. Forming $f(x)$ we have

$$f(x) \equiv \sum_{n=1}^{\infty} \left(\frac{1}{\lfloor n}\right)^2 (\sin \lfloor n \pi x) \sin\{\log(\sin^2 \lfloor n \pi x)\}$$

$$f'(x) \equiv \sum_{n=1}^{\infty} \left(\frac{\pi}{\lfloor n}\right)\{\cos \lfloor n \pi x [\sin\{\log(\sin^2 \lfloor n \pi x)\}]$$

$$+ \sin \lfloor n \pi x \cdot \cos\{\log(\sin^2 \lfloor n \pi x)\} \cdot 2$$

$$\sin \lfloor n \pi x \cos \lfloor n \pi x\}$$

$$\equiv \sum_{n=1}^{\infty} \frac{\pi}{\lfloor n}\left\{\cos \lfloor n \pi x [\sin\{\log(2\sin^2 \lfloor n \pi x)\}]\right.$$

$$\left. + 2\sin^2 \lfloor n \pi x \cdot \cos \lfloor n \pi x \{\cos\{\log(\sin^2 \lfloor n \pi x)\}\}\right\}$$

This expression is fixed when $\sin \lfloor n \pi x$ is not zero, hence $f(x)$ has a fixed

derivative at every irrational point. At the rational points, the summation from $n=1$ to $\alpha-1$ will be fixed and may be represented by g while on the rest of the terms, since

$$\sin^2 \lfloor n \rfloor \pi \tfrac{m}{q} = 0$$

the last term drops out and we have

$$f'\left(\tfrac{m}{q}\right) \equiv g + \sum_{1=\alpha}^{\infty} \frac{\pi}{\lfloor n \rfloor} \left\{ \cos \lfloor n \rfloor \pi \tfrac{m}{q} \left[\sin \left\{ \log \left(\sin^2 \lfloor n \rfloor \pi \tfrac{m}{q} \right) \right\} \right] \right\}$$

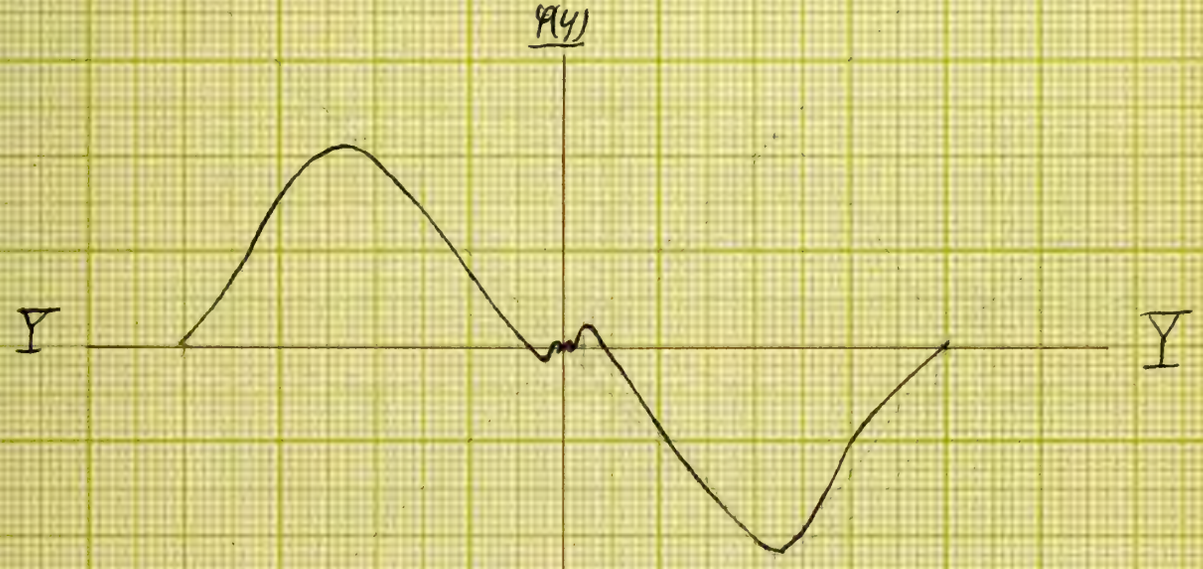
$$= g + \sum_{n=\alpha}^{\infty} \frac{\pi}{\lfloor n \rfloor} \left\{ \sin (\log 0) \right\}$$

$$= g + \sum_{n=\alpha}^{\infty} \frac{\pi}{n} \left\{ \sin \infty \right\}$$

which evidently oscillates and hence $f(x)$ has no derivative at any rational point.

Graph of
the function

$\varphi(y) \equiv |y|$



Sketch
Showing the behavior
of the function

$\varphi(y) \equiv y \sin(\log y^2)$
near the point
$y = 0$.

Scale - 4 Squares = 1 Unit.

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(3) Given: $\varphi(y) \equiv \frac{y}{1+e^{1/y}}$ (See page 169).

Examining the derivative of this function at the point $y=0$, we have

$$\frac{\varphi(y+\delta)-\varphi(y)}{\delta} = \frac{\frac{y+\delta}{1+e^{\frac{1}{y+\delta}}} - \frac{y}{1+e^{1/y}}}{\delta}$$

letting $y=0$, we have

$$\frac{\varphi(\delta)}{\delta} = \frac{\frac{\delta}{1+e^{1/\delta}}}{\delta} = \frac{1}{1+e^{1/\delta}}$$

and the limit of this expression is 0 as δ approaches zero. In like manner we find

$$\frac{\varphi(-\delta)}{\delta} = \frac{1}{1+e^{1/-\delta}}$$

and the limit of this expression is 1 as δ approaches zero. Therefore $\varphi(y)$ has no derivative at the point $y=0$. Forming our $f(x)$ now we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{\sin n!\,\pi x}{1+e^{\sin n!\,\pi x}} \cdot \left(\frac{1}{n!}\right)^2$$

Since $\varphi(y)$ has a fixed and finite derivative for every value of y between -1 and $+1$ except at zero, $f(x)$ will have a fixed and finite derivative at every irrational point. At every rational point we have, n

$$\frac{f\left(\frac{h}{k}\pm\delta\right)-f\left(\frac{h}{k}\right)}{\delta}=g+\sum_{n=\alpha}^{\infty}\left(\frac{1}{\lfloor n}\right)^2\frac{\pm\sin\lfloor n\,\delta\pi}{\delta}}{1+e^{\pm 2i\sin\lfloor n\,\pi\delta}}.$$

When k is even we have,

$$\lim_{\delta=0}\left(\frac{f\left(\frac{h}{k}+\delta\right)-f\left(\frac{h}{k}\right)}{\delta}\right)=g+\sum_{n=\alpha}^{\infty}\lim_{\delta=0}\left(\frac{1}{\lfloor n}\right)^2\frac{\sin\lfloor n\,\pi\delta}{\delta}}{1+e^{2i\sin\lfloor n\,\pi\delta}}$$

$$=g+C$$

$$\lim_{\delta=0}\frac{f\left(\frac{h}{k}-\delta\right)-f\left(\frac{h}{k}\right)}{-\delta}=g+\sum_{n=\alpha}^{\infty}\lim_{\delta=0}\left(\frac{1}{\lfloor n}\right)^2\left\{\frac{-\sin\lfloor n\,\pi\delta}{\delta}}{1+e^{2i\sin\lfloor n\,\pi\delta}}\right\}$$

$$=g+\sum_{n=\alpha}^{\infty}\frac{\pi}{\lfloor n}$$

Therefore $f(x)$ has no derivative in the rational points for which k is even.
When k is odd we have,

$$\frac{f\left(\frac{h}{k}+\delta\right)-f\left(\frac{h}{k}\right)}{\delta}=g+\left(\frac{1}{\lfloor\alpha}\right)^2\left\{\frac{-\sin\lfloor n\,\pi\delta}{\delta}}{1+e^{-2i\sin\lfloor n\,\pi\delta}}\right\}$$

$$+\sum_{n=\alpha+1}^{\infty}\left(\frac{1}{\lfloor n}\right)^2\left\{\frac{\sin\lfloor n\,\pi\delta}{\delta}}{1+e^{2i\sin\lfloor n\,\pi\delta}}\right\}$$

$$=g-\frac{\pi}{\lfloor\alpha}$$

in the limit.

Again

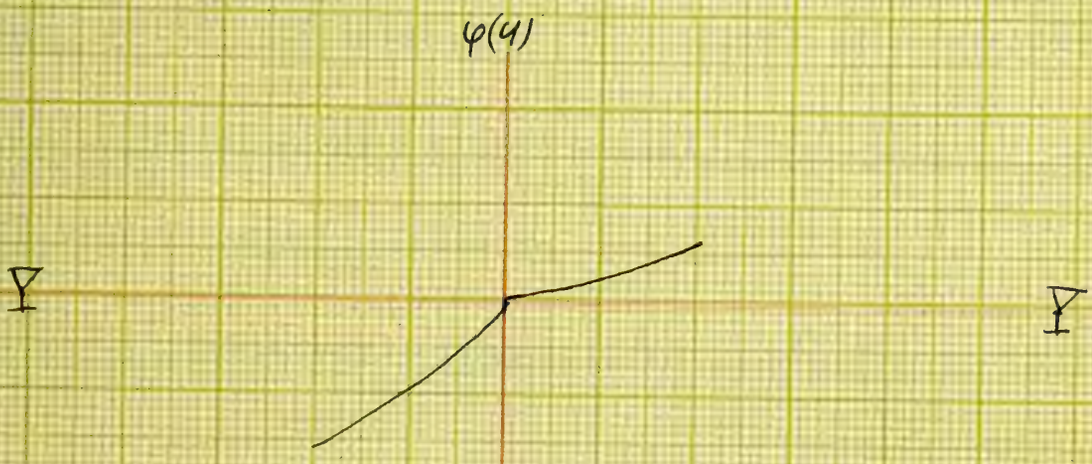
$$\frac{f\left(\frac{h}{k}-\delta\right)-f\left(\frac{h}{k}\right)}{-\delta}=g+\left(\frac{1}{\lfloor\alpha}\right)^2\left\{\frac{\frac{\sin\lfloor n\,\pi\delta}{-\delta}}{1+e^{2i\sin\lfloor n\,\pi\delta}}\right\}$$

$$+ \sum_{n=\alpha+1}^{\infty} \left(\frac{1}{n!}\right)^2 \left\{ \frac{-2\pi i n! s}{1 + e^{-2\pi i n! s}} \right\}$$

$$- f - \sum_{n=\alpha+1}^{\infty} \frac{1}{n!}$$

in the limit.

Therefore $f(x)$ has no derivative at any rational point.



Sketch of
the function

$$\varphi(4) = \frac{4}{1+e^{1/4}}$$

Scale - 2 Squares = 1 Unit.

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(4) Given;

$$\varphi(y) \equiv y - \tfrac{1}{2} y \sin\left(\frac{\log y^2}{2}\right)$$ (See page 164)

where we define $\varphi(0) = 0$. The derivative of this function is then,

$$\varphi'(y) \equiv 1 - \tfrac{1}{2}\left[\sin\left(\tfrac{1}{2}\log y^2\right) + y \cos\tfrac{1}{2}\log y^2 \cdot \tfrac{1}{y^2} \cdot 2y\right]$$

$$= 1 - \tfrac{1}{2}\sin\left(\tfrac{1}{2}\log y^2\right) - \cos\left(\tfrac{1}{2}\log y^2\right).$$

This expression becomes indeterminate when $y = 0$. Forming $f(x)$ now we have,

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^3}\left[\sin n\pi x - \tfrac{1}{2}\sin n\pi x \left\{\sin\left(\tfrac{1}{2}\log \sin^2 n\pi x\right)\right\}\right]$$

Taking the derivative, we have,

$$f'(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^3}\left[n\pi \cos n\pi x - \tfrac{1}{2}\left(n\pi \cos n\pi x \sin\left\{\tfrac{1}{2}\log \sin^2 n\pi x\right\}\right.\right.$$

$$\left.\left. \tfrac{1}{2}\pi n \sin n\pi x \cos\left(\log \sin^2 n\pi x\right) \frac{1}{\sin^2 n\pi x} \cdot 2\sin n\pi x \cos n\pi x\right)\right]$$

$$\equiv \sum_{n=1}^{\infty} \frac{\pi}{n^2}\left[\cos n\pi x - \tfrac{1}{2}\cos n\pi x \sin\left(\tfrac{1}{2}\log \sin^2 n\pi x\right)\right.$$

$$\left. - \cos n\pi x \cos\left(\tfrac{1}{2}\log \sin^2 n\pi x\right)\right]$$

$$\equiv \sum_{n=1}^{\infty} \frac{\pi}{n^2} \cos n\pi x \left[1 - \tfrac{1}{2}\sin\left(\tfrac{1}{2}\log \sin^2 n\pi x\right)\right.$$

$$\left. - \cos\left(\tfrac{1}{2}\log \sin^2 n\pi x\right)\right]$$

This expression is fixed and finite for

all irrational values of x. For the rational values we have,

$$f\left(\tfrac{h}{g}\right) \equiv \sum_{n=1}^{\infty} \frac{\pi}{n^2} \cos n\pi \tfrac{h}{g} \left[1 - \tfrac{1}{2} \sin\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right) - \cos\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right)\right] \quad \text{if } n \neq Kg$$

$$+ \sum_{n=1}^{\infty} \frac{\pi}{n^2} \cos n\pi \tfrac{h}{g} \left[1 - \tfrac{1}{2} \sin\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right) - \cos\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right)\right] \quad \text{if } n = Kg$$

$$= \sum_{n=1}^{\infty} \frac{\pi}{n^2} \cos n\pi \tfrac{h}{g} \left[1 - \tfrac{1}{2} \sin\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right) - \cos\left(\tfrac{1}{2} \log \sin^2 n\pi \tfrac{h}{g}\right)\right] \quad \text{if } n \neq Kg$$

$$+ \sum_{n=1}^{\infty} \frac{\pi}{n^2} \cos K\pi \left[1 - \tfrac{1}{2} \sin\left(\tfrac{1}{2} \log \sin^2 K\pi\right) - \cos\left(\tfrac{1}{2} \log \sin^2 K\pi\right)\right] \quad K = \frac{n}{g}$$

The first summation in this expression is fixed and finite, but the second is indeterminate, since $\sin K\pi = 0$. Therefore $f(x)$ has no fixed derivative at any rational point

Section XIII.

The problems given in this section are intended to illustrate the method set forth in section VII.

(1) Given;

$$\varphi(y) \equiv |y|. \qquad \text{(See page 164)}$$

(By reference to page given we see that $\varphi(y)$ is equal to 1 when y is positive and equal to -1 when y is negative. Forming $f(x)$, then, we have,

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{|n|} |x - a_n|$$

Taking the derivative,

$$f'(x) \equiv \sum_{n=1}^{k-1} \frac{1}{|n|} + \sum_{n=k+1}^{\infty} \frac{1}{|n|} + \frac{1}{\delta=0} \frac{|\delta|}{|k|}$$

Since this expression becomes, in the limit

$$f'(x) \equiv \sum_{n=1}^{k-1} \frac{1}{|n|} + \sum_{n=k+1}^{\infty} \frac{1}{|n|} \pm \frac{1}{|k|}.$$

Therefore $f'(x)$ is indeterminate when x is equal to any a_k of the given menge.

(2). Given

$$\varphi(y) = y^{2/3} \qquad \text{(See page (78)}$$

This is the curve called the semi-cubical parabola. It is continuous within any finite interval, and has a fixed and finite derivative at every point within that interval except at the point $y=0$ where it becomes infinite and of undetermined sign. This may be easily seen from the following;

$$\varphi'(y) \equiv \tfrac{2}{3}\, y^{-1/3}$$

$$= \frac{2}{3\sqrt[3]{y}}$$

This becomes $-\infty$ as y approaches zero thro negative values and $+\infty$ as y approaches thro positive values. Hence $\varphi'(y)$ is indeterminate in the point $y=0$. Forming $f(x)$ then we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} (x-w_n)^{2/3}$$

where w_n is any rational number. Taking the derivative we get

$$f'(x) \equiv \sum_{n=1}^{\infty} \frac{2/3}{n^2} (x-w_n)^{-1/3}$$

$$= \sum_{n=1}^{\infty} \frac{2}{3n^2} (x-w_n)^{-1/3}$$

which is determined for any irrational value of x, since $(x-w_n)$ cannot become

zero for any of these values of x. If any rational point, $x=u_k$, we have

$$f'(u_k) \equiv \sum_{n=1}^{k-1} \frac{2}{3n^2}(u_k - u_n)^{-1/3} + \sum_{n=k+1}^{\infty} \frac{2}{3n^2}(u_k - u_n) + \frac{2}{3k^2}(u_k - u_k)^{-1/3}$$

The first two parts of this expression are evidently fixed, but the last term, $\frac{2}{3k^2}(u_k - u_k)^{-1/3}$ evidently is indeterminate. Therefore $f(x)$ is indeterminate in every rational point.

(3). Given;

$$\varphi(y) \equiv \pm\sqrt{y\,|y|\,(a^2 - y^2)}$$

When y is positive we have

$$\varphi(y) \equiv \pm\sqrt{y^2(a^2-y^2)}$$

$$= \pm y\sqrt{a^2-y^2}.$$

When y is negative,

$$\varphi(y) \equiv \pm\sqrt{-y^2(a^2-y^2)}$$

which is imaginary when y is less than a. Therefore $\varphi(y)$ is imaginary for negative values of y. Taking the derivative we have

$$\varphi'(y) \equiv \pm\sqrt{a^2-y^2} \pm y\,\frac{-2y}{2\sqrt{a^2-y^2}}$$

$$= \pm\sqrt{a^2-y^2} \pm \frac{y^2}{\sqrt{a^2-y^2}}$$

When y is zero this becomes

$$\varphi(0) = \pm\sqrt{a^2} = \pm a$$

Therefore $\varphi(y)$ has no fixed derivative at the point $y=0$. Moreover the derivative of $\varphi(y)$ is finite for all values

of y which are less than a. Letting

$$c_n = 1/n$$

and w_n be a rational number, we have for $f(x)$

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \sqrt{(x-w_n)|x-w_n|(a^2-(x-w_n)^2)}$$

Since $\varphi(y)$ had a definite derivative for all points between 0 and a, we may expect $f(x)$ to have a definite derivative when

$$0 < |x-w_n| < a.$$

We know also that when

$$(x-w_n) > 0$$

we have

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \sqrt{(x-w_n)^2(a^2-(x-w_n)^2)}$$

Taking the derivative we have

$$f'(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \pm\sqrt{a^2-(x-w_n)^2} \pm \frac{(x-w_n)^2}{\sqrt{a^2-(x-w_n)^2}} \right\}$$

which is a fixed and finite quantity within the limits

$$0 < x-w_n < a.$$

When x is irrational then, $f(x)$ has a fixed derivative, provided that

$$(x - w_n) < a.$$

If x is a rational number then we have

$$f(w_k) = \sum_{n=1}^{k-1} \frac{1}{2^n} \left\{ \pm \sqrt{a^2 - (w_k - w_n)^2} + \frac{(w_k - w_n)^2}{\sqrt{a^2 - (w_k - w_n)^2}} \right\}$$

$$+ \frac{1}{2^k} \left\{ \pm \sqrt{a^2 - (w_k - w_k)^2} \pm \frac{(w_k - w_k)^2}{\sqrt{a^2 - (w_k - w_n)^2}} \right\}$$

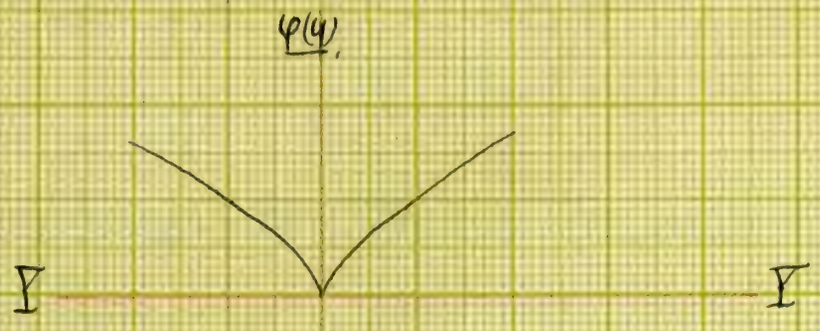
$$+ \sum_{n=k+1}^{\infty} \frac{1}{2^n} \left\{ \pm \sqrt{a^2 - (w_k - w_n)^2} \pm \frac{(w_k - w_n)^2}{\sqrt{a^2 - (w_k - w_n)^2}} \right\}$$

$$= \sum_{n=1}^{k-1} \frac{1}{2^n} \left\{ \pm \sqrt{a^2 - (w_k - w_n)^2} \pm \frac{(w_k - w_n)^2}{\sqrt{a^2 - (w_k - w_n)^2}} \right\}$$

$$+ \sum_{n=k+1}^{\infty} \frac{1}{2^n} \left\{ \pm \sqrt{a^2 - (w_k - w_n)^2} \pm \frac{(w_k - w_n)^2}{\sqrt{a^2 - (w_{k+} - w_n)^2}} \right\}$$

$$+ \frac{1}{2^k} (\pm \sqrt{a^2})$$

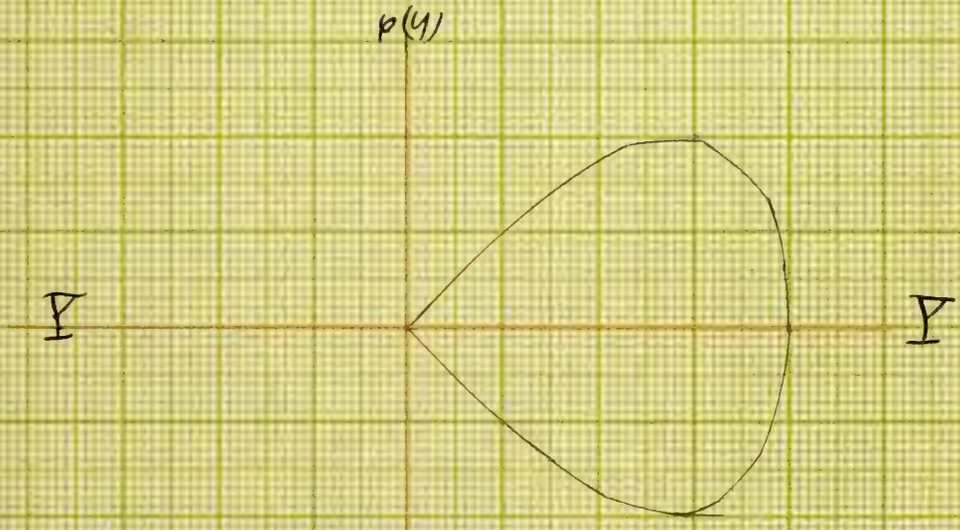
Since the last term in this expression has two values, the derivative of $f(x)$ does not exist at any rational point, within the limits given.



Sketch of
the function

$\varphi(y) \equiv y^{2/3}$

Scale- 1 Square = 1 Unit.



Sketch of the
function

$\varphi(y) \equiv \sqrt{y|y|(a^2-y^2)}$

$a \equiv 4$ Squares.

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List of functions treated by the various methods and reference to text where they may be found.

(1) $\varphi(y) \equiv \lim_{m=\infty} (y^2)^{1/m}$

Pages --- 23, 46, 50, 53, 140.

Found in

(2) $\varphi(y) \equiv \lim_{m=\infty} \{1-(\cos y)^{2m}\}$

Pages --- 38, 157.

Found in

(3) $\varphi(y) \equiv \frac{1}{1+e^{\pi y}}$

Pages --- 61, 120, 148.

Found in

(4) $\varphi(y) \equiv \lim_{t=0} \left(\frac{y^2}{y^2+t^2}\right)$

Pages 66, 68.

Found in Pascal - page 3.

(5) $\varphi(y) \equiv y \sin 1/y$

Pages --- 79, 100.

Found in Dini - page 184.

(6) $\varphi(y) \equiv \log \sin y$

Page --- 94.

Obtained by integrating $\cot y$.

(7) $\varphi(y) \equiv y \cos 1/y$

Page ---- 113.

Adapted from (5).

(8) $\varphi(y) \equiv \lim_{m=\infty} \left\{ \frac{(1+y^2)^m - 1}{(1+y^2)^m + 1} \right\}$

Page ----- 117.

Found in Pascal - page 29.

(9) $\varphi(y) \equiv \sin 1/y$

Page ---- 124.

Found in Dini - page 186.

(10) $\varphi(y) \equiv y + \sin 1/y + \sin|1/y|$

Pages ---- 126, 143.

Built up from (9).

(11) $\varphi(y) \equiv \frac{1}{1+e^{1/y}} + \sin 1/y + \sin|1/y|$

Page ------ 132.

Built up from (3) and (9).

(12) $\varphi(y) \equiv \frac{e^{1/y} - 1}{e^{1/y} + 1}$

Page ---- 151.

Found in Pascal - page 30.

(13) $\varphi(y) \equiv \frac{e^{1/y}}{1+e^{1/y}}$

Page ------ 154.

Adapted from (12).

(14) $\varphi(y) \equiv |y|$

Pages - 166, 172.

Adapted from $\varphi(y) \equiv \sqrt{y^2}$, Uiii - page 183.

(15) $\varphi(y) \equiv y \sin(\log y^2)$

Page - - - 162.

Found in Uiii - page 184.

(16) $\varphi(y) \equiv \frac{y}{1+e^{y^4}}$

Page - - - 165.

Found in Byerly - page 251.

(17) $\varphi(y) \equiv y - \frac{1}{2} y \sin(\frac{1}{2} - \varphi(y^2))$

Page - - - 170.

Found in Uiii - page 206.

(18) $\varphi(y) \equiv y^{2/3}$

Page - - - - 173.

Adapted from Taylor, - page 164, Ex. 1.

(19) $\varphi(y) \equiv \sqrt{y|y|(a^2 - y^2)}$

Page - - - 175.

Adapted from Taylor - page 162, Ex. 2.

Books used for reference in gathering materiál fr this thesis.

1. Dini - Theorie der Functionen Einer Veränderlichen Reellen Grösse.
2. Pascal - Esercizi e Note Critiche di Calcolo Infinitesimale.
3. Taylor - Elements of Calculus.
4. Byerly - Differential Calculus.
5. Encyklopädie der Mathematischen Wissenschaften. Teil I, Band II, Heft I.

Books in which phases of this subject are treated, but not used in the preparation of this thesis.

1. Lit. Centr-Bl 1871 - p. 150.
2. Math. Ann. 18. (1882) p. 588.
3. " " 47. (1896) p. 298.

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